AA HL Practice Set 3 Paper 2 Solution

Section A

1. (a) (i) 6 A1

(ii) 6 A1

(iii) The range

$=18-3$ (M1) for valid approach

$=15$ A1

[4]

(b) (i) The mean

$$=\frac{(3)(12)+(6)(20)+(9)(12)+(12)(8)+(15)(4)+(18)(4)}{12+20+12+8+4+4}$$ (M1) for valid approach

$=8.2$ A1

(ii) The variance

$=4.308131846^2$ (M1) for valid approach

$=18.6$ A1

[4]

2. (a) $f(x)=g(x)$

$\pi e^{-x^2}=1+\dfrac{1}{\pi e^{-x^2}}$ (M1) for setting equation

$\pi e^{-x^2}-1-\dfrac{e^{x^2}}{\pi}=0$

By considering the graph of $y=\pi e^{-x^2}-1-\dfrac{e^{x^2}}{\pi}$,

$x=-0.814566$ or $x=0.8145662$.

$\therefore a=-0.815$, $b=0.815$ A2

[3]

(b) The required area

$=\int_{-0.814566}^{0.8145662}(f(x)-g(x))\mathrm{d}x$ (A1) for correct integral

$=\int_{-0.814566}^{0.8145662}\left(\pi e^{-x^2}-1-\dfrac{e^{x^2}}{\pi}\right)\mathrm{d}x$

$=1.890606422$ (A1) for correct value

$=1.89$ A1

[3]

3. Note that $f(0)=-1$.

$-1=\sqrt{2}\sin\left(\dfrac{\pi}{6}(0+h)\right)$ (M1) for setting equation

$-\dfrac{1}{\sqrt{2}}=\sin\left(\dfrac{\pi}{6}h\right)$ (A1) for correct approach

$\dfrac{\pi}{6}h=-\dfrac{3\pi}{4}$ or $\dfrac{\pi}{6}h=-\dfrac{\pi}{4}$ (A1) for correct approach

$h=-4.5$ (*Rejected*) or $h=-1.5$ A1

$\therefore h=-1.5$ A1

[5]

4. (a) (i) $\frac{1}{2}$ A1

(ii) 3 A1

(iii) –4 A1

[3]

(b) The coordinates of P'

$=\left(\frac{2}{2}+3,-5(8-4)\right)$ (A2) for correct approach

$=(4,-20)$ A2

[4]

5. (a) $\cos\theta=\frac{AB}{r}$

$AB=r\cos\theta$ A1

[1]

(b) $\sin\theta=\frac{AE}{r}$

$AE=r\sin\theta$ A1

The area of the triangle ABE

$=\frac{(AB)(AE)}{2}$

$=\frac{(r\cos\theta)(r\sin\theta)}{2}$ M1

$=\frac{1}{2}r^2\sin\theta\cos\theta$ A1

$=\frac{1}{2}r^2\left(\frac{1}{2}\sin 2\theta\right)$ A1

$=\frac{r^2\sin 2\theta}{4}$ AG

[4]

(c) $A\hat{E}B+B\hat{E}C+C\hat{E}D=\pi$ M1

$\left(\frac{\pi}{2}-\theta\right)+B\hat{E}C+\left(\frac{\pi}{2}-\theta\right)=\pi$ A1

$\pi-2\theta+B\hat{E}C=\pi$

$B\hat{E}C=2\theta$ AG

[2]

6. (a) Let $\frac{x^2+2x+4}{(x-3)(x-7)} \equiv A+\frac{B}{x-3}+\frac{C}{x-7}$, where A, B and C are constants.

$\frac{x^2+2x+4}{(x-3)(x-7)} \equiv \frac{A(x-3)(x-7)}{(x-3)(x-7)}$ M1

$+\frac{B(x-7)}{(x-3)(x-7)}+\frac{C(x-3)}{(x-3)(x-7)}$

$\frac{x^2+2x+4}{(x-3)(x-7)}$

$\equiv \frac{Ax^2-10Ax+21A+Bx-7B+Cx-3C}{(x-3)(x-7)}$

x^2+2x+4

$\equiv Ax^2+(-10A+B+C)x+(21A-7B-3C)$ A1

$A=1$ A1

$2=-10(1)+B+C$

$C=12-B$

$4=21A-7B-3C$

$\therefore 4=21(1)-7B-3(12-B)$ A1

$4=21-7B-36+3B$

$19=-4B$

$B=-\frac{19}{4}$ A1

$\therefore C=12-\left(-\frac{19}{4}\right)$

$C=\frac{67}{4}$ A1

[6]

(b) $y=1$ A1

[1]

7. (a) (i) $\begin{cases} x+2y-z=1 \\ 2x-y+az=0 \\ x+3y+2z=b \end{cases}$

$\to \begin{cases} x+2y-z=1 \\ -5y+(a+2)z=-2 \\ y+3z=b-1 \end{cases}$ M1

$(R_2-2R_1 \;\&\; R_3-R_1)$

$\to \begin{cases} x+2y-z=1 \\ -5y+(a+2)z=-2 \quad (R_3+0.2R_2) \\ (0.2a+3.4)z=b-1.4 \end{cases}$ A1

The system has no solutions when

$0.2a+3.4=0$ and $b-1.4\neq 0$.

$a=-17$ and $b\neq 1.4$ A1

(ii) The system has a unique solution when

$0.2a+3.4\neq 0$.

$\therefore a\neq -17$ and $b\in\mathbb{R}$ A1

[4]

(b) $\begin{cases} x+2y-z=1 \\ 2x-y+3z=0 \\ x+3y+2z=3 \end{cases}$

By solving the system, $x=-0.2$, $y=0.8$ and

$z=0.4$. A2

[2]

8. $\mathbf{r}=(-1+2\lambda+4\mu)\mathbf{i}+(3+\lambda)\mathbf{j}+(-1+5\mu)\mathbf{k}$

$\mathbf{r}=(-\mathbf{i}+3\mathbf{j}-\mathbf{k})+\lambda(2\mathbf{i}+\mathbf{j})+\mu(4\mathbf{i}+5\mathbf{k})$ (M1) for valid approach

$\mathbf{n}=(2\mathbf{i}+\mathbf{j})\times(4\mathbf{i}+5\mathbf{k})$

$\mathbf{n}=\begin{pmatrix} (1)(5)-(0)(0) \\ (0)(4)-(2)(5) \\ (2)(0)-(1)(4) \end{pmatrix}$

$\mathbf{n}=5\mathbf{i}-10\mathbf{j}-4\mathbf{k}$ (A1) for correct values

The Cartesian equation of the plane π:

$(x\mathbf{i}+y\mathbf{j}+z\mathbf{k})\cdot(5\mathbf{i}-10\mathbf{j}-4\mathbf{k})=(-\mathbf{i}+3\mathbf{j}-\mathbf{k})\cdot(5\mathbf{i}-10\mathbf{j}-4\mathbf{k})$ M1A1

$5x-10y-4z=(-1)(5)+(3)(-10)+(-1)(-4)$

$5x-10y-4z=-31$ A1

[5]

9. (a) $f(0) = \arctan\frac{\pi}{2}(0) = 0$ (A1) for correct value

$$f'(x) = \left(\frac{1}{1+\left(\frac{\pi}{2}x\right)^2}\right)\left(\frac{\pi}{2}\right)$$

$$f'(x) = \frac{2\pi}{4+\pi^2x^2}$$

$$f'(0) = \frac{2\pi}{4+\pi^2(0)^2} = \frac{\pi}{2}$$ (A1) for correct value

$$f''(x) = \frac{(4+\pi^2x^2)(0)-(2\pi)(2\pi^2x)}{(4+\pi^2x^2)^2}$$ (M1) for valid approach

$$f''(x) = -\frac{4\pi^3x}{(4+\pi^2x^2)^2}$$

$$f''(0) = -\frac{4\pi^3(0)}{(4+\pi^2(0)^2)^2} = 0$$ (A1) for correct value

$$f^{(3)}(x) = -\frac{(4+\pi^2x^2)^2(4\pi^3) - (4\pi^3x)(2)(4+\pi^2x^2)(2\pi^2x)}{(4+\pi^2x^2)^4}$$ (M1) for valid approach

$$f^{(3)}(0) = -\frac{(4+0)^2(4\pi^3)-0}{4^4} = -\frac{\pi^3}{4}$$ (A1) for correct value

$$f(x) = f(0) + xf'(0) + \frac{x^2}{2!}f''(0) + \frac{x^3}{3!}f^{(3)}(0) + \cdots$$

$$f(x) = 0 + x\left(\frac{\pi}{2}\right) + \frac{x^2}{2}(0) + \frac{x^3}{6}\left(-\frac{\pi^3}{4}\right) + \cdots$$

$$f(x) = \frac{\pi}{2}x - \frac{\pi^3}{24}x^3 + \cdots$$ A1

[7]

Section B

10. (a) $a = -0.0807147258$

$a = -0.0807$ A1

$b = 3.177202711$

$b = 3.18$ A1

[2]

(b) $\log y = -0.0807147258\sqrt{9} + 3.177202711$ (M1) for valid approach

$\log y = 2.935058534$

$y = 10^{2.935058534}$ (M1) for valid approach

$y = 861.1098035$

$y = 861$ A1

[3]

(c) $\log y = -0.0807147258\sqrt{x} + 3.177202711$

$y = 10^{-0.0807147258\sqrt{x}+3.177202711}$ (M1) for valid approach

$y = 10^{-0.0807147258\sqrt{x}} \cdot 10^{3.177202711}$ (A1) for correct approach

$y = 10^{3.177202711} \cdot (10^{-0.0807147258})^{\sqrt{x}}$ A1

$k = 10^{3.177202711}$ (A1) for correct approach

$k = 1503.843735$

$k = 1500$ A1

$m = 10^{-0.0807147258}$ (A1) for correct approach

$m = 0.8303960491$

$m = 0.830$ A1

[7]

11. (a) $a=\dfrac{v^2+64}{240}$

$\dfrac{dv}{dt}=\dfrac{v^2+64}{240}$

$\dfrac{1}{v^2+64}dv=\dfrac{1}{240}dt$ (M1) for valid approach

$\int\dfrac{1}{v^2+64}dv=\int\dfrac{1}{240}dt$ (A1) for correct approach

$\dfrac{1}{8}\arctan\dfrac{v}{8}=\dfrac{1}{240}t+C$ A1

$\arctan\dfrac{v}{8}=\dfrac{1}{30}t+C$

$\dfrac{v}{8}=\tan\left(\dfrac{1}{30}t+C\right)$

$v=8\tan\left(\dfrac{1}{30}t+C\right)$ A1

$0=8\tan\left(\dfrac{1}{30}(0)+C\right)$ (M1) for substitution

$C=0$ (A1) for correct value

$\therefore v=8\tan\dfrac{1}{30}t$ A1

[7]

(b) $\arctan\dfrac{v}{8}=\dfrac{1}{30}t$

$\arctan\left(\dfrac{1}{8}\cdot\dfrac{8}{3}\sqrt{3}\right)=\dfrac{1}{30}t$ (M1) for setting equation

$\arctan\dfrac{\sqrt{3}}{3}=\dfrac{1}{30}t$

$\dfrac{\pi}{6}=\dfrac{1}{30}t$ (A1) for correct approach

$t=5\pi$ s A1

[3]

(c)	$\dfrac{\mathrm{d}v}{\mathrm{d}t}=\dfrac{v^2+64}{240}$	
	$\dfrac{\mathrm{d}v}{\mathrm{d}s}\cdot\dfrac{\mathrm{d}s}{\mathrm{d}t}=\dfrac{v^2+64}{240}$	A1
	$v\dfrac{\mathrm{d}v}{\mathrm{d}s}=\dfrac{v^2+64}{240}$	A1
	$\dfrac{240v}{v^2+64}\mathrm{d}v=\mathrm{d}s$	M1
	$\int\dfrac{240v}{v^2+64}\mathrm{d}v=\int\mathrm{d}s$	A1
	$s=\int\dfrac{240v}{v^2+64}\mathrm{d}v$	AG
		[4]
(d)	$s=\int\dfrac{240v}{v^2+64}\mathrm{d}v$	
	Let $u=v^2+64$.	(M1) for substitution
	$\dfrac{\mathrm{d}u}{\mathrm{d}v}=2v\Rightarrow 240v\mathrm{d}v=120\mathrm{d}u$	
	$\therefore s=\int\dfrac{1}{u}\cdot 120\mathrm{d}u$	A1
	$s=120\ln\lvert u\rvert+D$	
	$s=120\ln(v^2+64)+D$	A1
	$0=120\ln(0^2+64)+D$	(M1) for substitution
	$D=-120\ln 64$	(A1) for correct value
	$\therefore s=120\ln(v^2+64)-120\ln 64$	
	$s=120\ln\left(\left(\dfrac{8}{3}\sqrt{3}\right)^2+64\right)-120\ln 64$	
	$s=34.52184869\text{ m}$	
	$s=34.5\text{ m}$	A1
		[6]

12. (a) $\left(\cos\frac{\theta}{7}+i\sin\frac{\theta}{7}\right)^7$

$=\cos^7\frac{\theta}{7}+\binom{7}{1}i\cos^6\frac{\theta}{7}\sin\frac{\theta}{7}+\binom{7}{2}i^2\cos^5\frac{\theta}{7}\sin^2\frac{\theta}{7}$

$+\binom{7}{3}i^3\cos^4\frac{\theta}{7}\sin^3\frac{\theta}{7}+\binom{7}{4}i^4\cos^3\frac{\theta}{7}\sin^4\frac{\theta}{7}$

$+\binom{7}{5}i^5\cos^2\frac{\theta}{7}\sin^5\frac{\theta}{7}+\binom{7}{6}i^6\cos\frac{\theta}{7}\sin^6\frac{\theta}{7}$ A2

$+i^7\sin^7\frac{\theta}{7}$

$=\cos^7\frac{\theta}{7}+7i\cos^6\frac{\theta}{7}\sin\frac{\theta}{7}-21\cos^5\frac{\theta}{7}\sin^2\frac{\theta}{7}$

$-35i\cos^4\frac{\theta}{7}\sin^3\frac{\theta}{7}+35\cos^3\frac{\theta}{7}\sin^4\frac{\theta}{7}$ A1

$+21i\cos^2\frac{\theta}{7}\sin^5\frac{\theta}{7}-7\cos\frac{\theta}{7}\sin^6\frac{\theta}{7}-i\sin^7\frac{\theta}{7}$

$\therefore \cos\theta+i\sin\theta$

$=\cos^7\frac{\theta}{7}+7i\cos^6\frac{\theta}{7}\sin\frac{\theta}{7}-21\cos^5\frac{\theta}{7}\sin^2\frac{\theta}{7}$

$-35i\cos^4\frac{\theta}{7}\sin^3\frac{\theta}{7}+35\cos^3\frac{\theta}{7}\sin^4\frac{\theta}{7}$ M1

$+21i\cos^2\frac{\theta}{7}\sin^5\frac{\theta}{7}-7\cos\frac{\theta}{7}\sin^6\frac{\theta}{7}-i\sin^7\frac{\theta}{7}$

$=\cos^7\frac{\theta}{7}-21\cos^5\frac{\theta}{7}\sin^2\frac{\theta}{7}$

$+35\cos^3\frac{\theta}{7}\sin^4\frac{\theta}{7}-7\cos\frac{\theta}{7}\sin^6\frac{\theta}{7}$

$+i\begin{pmatrix}7\cos^6\frac{\theta}{7}\sin\frac{\theta}{7}-35\cos^4\frac{\theta}{7}\sin^3\frac{\theta}{7}\\+21\cos^2\frac{\theta}{7}\sin^5\frac{\theta}{7}-\sin^7\frac{\theta}{7}\end{pmatrix}$

$\therefore \cos\theta=\cos^7\frac{\theta}{7}-21\cos^5\frac{\theta}{7}\sin^2\frac{\theta}{7}$

$+35\cos^3\frac{\theta}{7}\sin^4\frac{\theta}{7}-7\cos\frac{\theta}{7}\sin^6\frac{\theta}{7}$ and

$\sin\theta=7\cos^6\frac{\theta}{7}\sin\frac{\theta}{7}-35\cos^4\frac{\theta}{7}\sin^3\frac{\theta}{7}$

$+21\cos^2\frac{\theta}{7}\sin^5\frac{\theta}{7}-\sin^7\frac{\theta}{7}$. A2

[6]

(b) $\tan\theta$

$= \dfrac{\sin\theta}{\cos\theta}$

$$= \frac{7\cos^6\frac{\theta}{7}\sin\frac{\theta}{7} - 35\cos^4\frac{\theta}{7}\sin^3\frac{\theta}{7} + 21\cos^2\frac{\theta}{7}\sin^5\frac{\theta}{7} - \sin^7\frac{\theta}{7}}{\cos^7\frac{\theta}{7} - 21\cos^5\frac{\theta}{7}\sin^2\frac{\theta}{7} + 35\cos^3\frac{\theta}{7}\sin^4\frac{\theta}{7} - 7\cos\frac{\theta}{7}\sin^6\frac{\theta}{7}}$$ M1A1

$$= \frac{7\tan\frac{\theta}{7} - 35\tan^3\frac{\theta}{7} + 21\tan^5\frac{\theta}{7} - \tan^7\frac{\theta}{7}}{1 - 21\tan^2\frac{\theta}{7} + 35\tan^4\frac{\theta}{7} - 7\tan^6\frac{\theta}{7}}$$ A1

Let $x = \tan\dfrac{\theta}{7}$.

$\tan\theta = \dfrac{7x - 35x^3 + 21x^5 - x^7}{1 - 21x^2 + 35x^4 - 7x^6}$ M1

$x^6 - 21x^4 + 35x^2 - 7 = 0$

$\dfrac{-x(x^6 - 21x^4 + 35x^2 - 7)}{1 - 21x^2 + 35x^4 - 7x^6} = 0$

$\dfrac{7x - 35x^3 + 21x^5 - x^7}{1 - 21x^2 + 35x^4 - 7x^6} = 0$

$\tan\theta = 0$ M1

$\theta = 0$, π, 2π, 3π, 4π, 5π or 6π

$\therefore x = \tan\dfrac{0}{7}$, $x = \tan\dfrac{\pi}{7}$, $x = \tan\dfrac{2\pi}{7}$, $x = \tan\dfrac{3\pi}{7}$,

$x = \tan\dfrac{4\pi}{7}$, $x = \tan\dfrac{5\pi}{7}$ or $x = \tan\dfrac{6\pi}{7}$ A1

$x = 0$ (*Rejected*), $x = \tan\dfrac{\pi}{7}$, $x = \tan\dfrac{2\pi}{7}$,

$x = \tan\dfrac{3\pi}{7}$, $x = \tan\dfrac{4\pi}{7}$, $x = \tan\dfrac{5\pi}{7}$ or $x = \tan\dfrac{6\pi}{7}$ A1

Thus, the equation $x^6 - 21x^4 + 35x^2 - 7 = 0$ has six roots. AG

[7]

(c) (i) $\sum_{r=1}^{7} \tan \frac{r\pi}{7}$

$= \sum_{r=1}^{6} \tan \frac{r\pi}{7} + \tan \frac{7\pi}{7}$ M1

$= -\frac{0}{1} + 0$ A1

$= 0$ A1

(ii) $\left(\tan \frac{\pi}{7}\right)\left(\tan \frac{2\pi}{7}\right)\left(\tan \frac{3\pi}{7}\right)\left(\tan \frac{4\pi}{7}\right)\left(\tan \frac{5\pi}{7}\right)\left(\tan \frac{6\pi}{7}\right) = -7$ M1A1

$\left(\tan \frac{\pi}{7}\right)\left(\tan \frac{2\pi}{7}\right)\left(\tan \frac{3\pi}{7}\right)\left(\tan\left(\pi - \frac{3\pi}{7}\right)\right)\left(\tan\left(\pi - \frac{2\pi}{7}\right)\right)\left(\tan\left(\pi - \frac{\pi}{7}\right)\right) = -7$

$\left(\tan \frac{\pi}{7}\right)\left(\tan \frac{2\pi}{7}\right)\left(\tan \frac{3\pi}{7}\right)\left(-\tan \frac{3\pi}{7}\right)\left(-\tan \frac{2\pi}{7}\right)\left(-\tan \frac{\pi}{7}\right) = -7$ A1

$\left(\tan \frac{\pi}{7} \tan \frac{2\pi}{7} \tan \frac{3\pi}{7}\right)^2 = 7$

$\therefore \tan \frac{\pi}{7} \tan \frac{2\pi}{7} \tan \frac{3\pi}{7} = \sqrt{7}$ A1

[7]