Chapter 10 Solution

Exercise 40

1. (a) $(x+2+h)^2=(x+2+h)(x+2+h)$

$(x+2+h)^2$

$=x^2+2x+hx+2x+4+2h+hx+2h+h^2$ (M1) for valid expansion

$(x+2+h)^2=x^2+(4+2h)x+(4+4h+h^2)$ A1

[2]

(b) $\frac{dy}{dx}=\lim_{h\to 0}\frac{(x+2+h)^2-(x+2)^2}{h}$

$\frac{dy}{dx}=\lim_{h\to 0}\frac{x^2+(4+2h)x+(4+4h+h^2)-(x^2+4x+4)}{h}$ (A1) for substitution

$\frac{dy}{dx}=\lim_{h\to 0}\frac{x^2+4x+2hx+4+4h+h^2-x^2-4x-4}{h}$

$\frac{dy}{dx}=\lim_{h\to 0}\frac{2hx+4h+h^2}{h}$

$\frac{dy}{dx}=\lim_{h\to 0}(2x+4+h)$ A1

$\frac{dy}{dx}=2x+4+0$

$\frac{dy}{dx}=2x+4$ A1

[3]

2. $f'(x)=\lim_{h\to 0}\dfrac{((x+h)^2-2(x+h)^3)-(x^2-2x^3)}{h}$ (A1) for substitution

$f'(x)=\lim_{h\to 0}\dfrac{\begin{pmatrix}x^2+2hx+h^2\\ -2\left(x^3+\binom{3}{1}x^2h+\binom{3}{2}xh^2+h^3\right)\end{pmatrix}-(x^2-2x^3)}{h}$ (M1) for valid expansion

$f'(x)=\lim_{h\to 0}\dfrac{\begin{pmatrix}x^2+2hx+h^2\\ -2(x^3+3x^2h+3xh^2+h^3)\end{pmatrix}-x^2+2x^3}{h}$

$f'(x)=\lim_{h\to 0}\dfrac{\begin{matrix}x^2+2hx+h^2-2x^3-6x^2h\\ -6xh^2-2h^3-x^2+2x^3\end{matrix}}{h}$

$f'(x)=\lim_{h\to 0}\dfrac{2hx+h^2-6x^2h-6xh^2-2h^3}{h}$ A1

$f'(x)=\lim_{h\to 0}(2x+h-6x^2-6xh-2h^2)$ A1

$f'(x)=2x+0-6x^2-6x(0)-2(0)$

$f'(x)=2x-6x^2$ A1

[5]

3. $f'(x)=\lim_{h\to 0}\dfrac{\dfrac{1}{3(x+h)-2}-\dfrac{1}{3x-2}}{h}$ (A1) for substitution

$f'(x)=\lim_{h\to 0}\dfrac{\dfrac{1}{3x+3h-2}-\dfrac{1}{3x-2}}{h}$

$f'(x)=\lim_{h\to 0}\dfrac{1}{h}\left(\dfrac{3x-2}{(3x+3h-2)(3x-2)}-\dfrac{3x+3h-2}{(3x+3h-2)(3x-2)}\right)$ (M1) for valid approach

$f'(x)=\lim_{h\to 0}\dfrac{1}{h}\left(\dfrac{3x-2-3x-3h+2}{(3x+3h-2)(3x-2)}\right)$

$f'(x)=\lim_{h\to 0}\dfrac{1}{h}\left(\dfrac{-3h}{(3x+3h-2)(3x-2)}\right)$ A1

$f'(x)=\lim_{h\to 0}\left(\dfrac{-3}{(3x+3h-2)(3x-2)}\right)$ A1

$f'(x)=\dfrac{-3}{(3x+3(0)-2)(3x-2)}$

$f'(x)=-\dfrac{3}{(3x-2)^2}$ A1

[5]

4. $$f'(x)=\lim_{h\to 0}\frac{\frac{1}{(1-2(x+h))^2}-\frac{1}{(1-2x)^2}}{h}$$ (A1) for substitution

$$f'(x)=\lim_{h\to 0}\frac{\frac{1}{(1-2x-2h)^2}-\frac{1}{(1-2x)^2}}{h}$$

$$f'(x)=\lim_{h\to 0}\frac{1}{h}\left(\frac{(1-2x)^2}{(1-2x-2h)^2(1-2x)^2}-\frac{(1-2x-2h)^2}{(1-2x-2h)^2(1-2x)^2}\right)$$ (M1) for valid approach

$$f'(x)=\lim_{h\to 0}\frac{1}{h}\left(\frac{(1-2x)^2-(1-2x-2h)^2}{(1-2x-2h)^2(1-2x)^2}\right)$$

$$f'(x)=\lim_{h\to 0}\frac{1}{h}\left(\frac{(1-2x+1-2x-2h)(1-2x-(1-2x-2h))}{(1-2x-2h)^2(1-2x)^2}\right)$$ A1

$$f'(x)=\lim_{h\to 0}\frac{1}{h}\left(\frac{(2-4x-2h)(2h)}{(1-2x-2h)^2(1-2x)^2}\right)$$

$$f'(x)=\lim_{h\to 0}\frac{4-8x-4h}{(1-2x-2h)^2(1-2x)^2}$$ A1

$$f'(x)=\frac{4(1-2x-0)}{(1-2x-2(0))^2(1-2x)^2}$$

$$f'(x)=\frac{4(1-2x)}{(1-2x)^4}$$

$$f'(x)=\frac{4}{(1-2x)^3}$$ A1

[5]

Exercise 41

1. Let $y = \arctan ex$.

$\tan y = ex$	A1
$\frac{\mathrm{d}}{\mathrm{d}x}(\tan y) = \frac{\mathrm{d}}{\mathrm{d}x}(ex)$	
$\frac{\mathrm{d}(\tan y)}{\mathrm{d}y} \cdot \frac{\mathrm{d}y}{\mathrm{d}x} = \frac{\mathrm{d}(ex)}{\mathrm{d}x}$	A1
$\sec^2 y \frac{\mathrm{d}y}{\mathrm{d}x} = e$	A1
$\frac{\mathrm{d}y}{\mathrm{d}x} = \frac{e}{\sec^2 y}$	M1
$\therefore \frac{\mathrm{d}}{\mathrm{d}x}(\arctan ex) = \frac{e}{1+\tan^2 y}$	A1
$\frac{\mathrm{d}}{\mathrm{d}x}(\arctan ex) = \frac{e}{1+(ex)^2}$	
$\frac{\mathrm{d}}{\mathrm{d}x}(\arctan ex) = \frac{e}{1+e^2x^2}$	AG

[5]

2. Let $y = \arccos\frac{x}{3}$.

$\cos y = \frac{x}{3}$ A1

$\frac{\mathrm{d}}{\mathrm{d}x}(\cos y) = \frac{\mathrm{d}}{\mathrm{d}x}\left(\frac{x}{3}\right)$

$\frac{\mathrm{d}(\cos y)}{\mathrm{d}y}\cdot\frac{\mathrm{d}y}{\mathrm{d}x} = \frac{\mathrm{d}}{\mathrm{d}x}\left(\frac{x}{3}\right)$ A1

$-\sin y\frac{\mathrm{d}y}{\mathrm{d}x} = \frac{1}{3}$ A1

$\frac{\mathrm{d}y}{\mathrm{d}x} = -\frac{1}{3\sin y}$ M1

$\therefore \frac{\mathrm{d}}{\mathrm{d}x}\left(\arccos\frac{x}{3}\right) = -\frac{1}{3\sqrt{1-\cos^2 y}}$ A1

$\frac{\mathrm{d}}{\mathrm{d}x}\left(\arccos\frac{x}{3}\right) = -\frac{1}{3\sqrt{1-\left(\frac{x}{3}\right)^2}}$

$\frac{\mathrm{d}}{\mathrm{d}x}\left(\arccos\frac{x}{3}\right) = -\frac{1}{\sqrt{9\left(1-\frac{x^2}{9}\right)}}$

$\frac{\mathrm{d}}{\mathrm{d}x}\left(\arccos\frac{x}{3}\right) = -\frac{1}{\sqrt{9-x^2}}$ AG

[5]

3. Let $y=\arcsin x^3$.

$\sin y = x^3$ A1

$\frac{\mathrm{d}}{\mathrm{d}x}(\sin y)=\frac{\mathrm{d}}{\mathrm{d}x}(x^3)$

$\frac{\mathrm{d}(\sin y)}{\mathrm{d}y}\cdot\frac{\mathrm{d}y}{\mathrm{d}x}=\frac{\mathrm{d}(x^3)}{\mathrm{d}x}$ A1

$\cos y\frac{\mathrm{d}y}{\mathrm{d}x}=3x^2$ A1

$\frac{\mathrm{d}y}{\mathrm{d}x}=\frac{3x^2}{\cos y}$ M1

$\therefore\frac{\mathrm{d}}{\mathrm{d}x}(\arcsin x^3)=\frac{3x^2}{\sqrt{1-\sin^2 y}}$ A1

$\frac{\mathrm{d}}{\mathrm{d}x}(\arcsin x^3)=\frac{3x^2}{\sqrt{1-(x^3)^2}}$

$\frac{\mathrm{d}}{\mathrm{d}x}(\arcsin x^3)=\frac{3x^2}{\sqrt{1-x^6}}$ AG

[5]

4. Let $y=\arctan\sqrt{x}$.

$\tan y=\sqrt{x}$ A1

$\frac{\mathrm{d}}{\mathrm{d}x}(\tan y)=\frac{\mathrm{d}}{\mathrm{d}x}(\sqrt{x})$

$\frac{\mathrm{d}(\tan y)}{\mathrm{d}y}\cdot\frac{\mathrm{d}y}{\mathrm{d}x}=\frac{\mathrm{d}(\sqrt{x})}{\mathrm{d}x}$ A1

$\sec^2 y\frac{\mathrm{d}y}{\mathrm{d}x}=\frac{1}{2\sqrt{x}}$ A1

$\frac{\mathrm{d}y}{\mathrm{d}x}=\frac{1}{2\sqrt{x}\sec^2 y}$ M1

$\therefore\frac{\mathrm{d}}{\mathrm{d}x}(\arctan\sqrt{x})=\frac{1}{2\sqrt{x}(1+\tan^2 y)}$ A1

$\frac{\mathrm{d}}{\mathrm{d}x}(\arctan\sqrt{x})=\frac{1}{2\sqrt{x}(1+(\sqrt{x})^2)}$

$\frac{\mathrm{d}}{\mathrm{d}x}(\arctan\sqrt{x})=\frac{1}{2\sqrt{x}(1+x)}$ AG

[5]

Exercise 42

1. When $n = 1$,

$$\text{L.H.S.} = -1\left(\frac{1}{(1+5x)^2}\right)(5)$$

$$\text{L.H.S.} = -\frac{5}{(1+5x)^2}$$

$$\text{R.H.S.} = \frac{(-5)^1 1!}{(1+5x)^{1+1}}$$

$$\text{R.H.S.} = -\frac{5}{(1+5x)^2}$$

Thus, the statement is true when $n = 1$. R1

Assume that the statement is true when $n = k$. M1

$$f^{(k)}(x) = \frac{(-5)^k k!}{(1+5x)^{k+1}}$$

When $n = k+1$,

$$f^{(k+1)}(x) = \frac{\mathrm{d}}{\mathrm{d}x}(f^{(k)}(x))$$ M1

$$f^{(k+1)}(x) = (-5)^k k!(-(k+1))\left(\frac{1}{(1+5x)^{k+2}}\right)(5)$$ A1

$$f^{(k+1)}(x) = \frac{(-5)^k(-5)(k+1)k!}{(1+5x)^{k+2}}$$ A1

$$f^{(k+1)}(x) = \frac{(-5)^{k+1}(k+1)!}{(1+5x)^{k+2}}$$

$$f^{(k+1)}(x) = \frac{(-5)^{k+1}(k+1)!}{(1+5x)^{k+1+1}}$$ A1

Thus, the statement is true when $n = k+1$.

Therefore, the statement is true for all $n \in \mathbb{Z}^+$. R1

[7]

2. When $n=1$,

$\text{L.H.S.}=(2x)(e^x)+(x^2)(e^x)$

$\text{L.H.S.}=2xe^x+x^2e^x$

$\text{R.H.S.}=(1(1-1)+2(1)x+x^2)e^x$

$\text{R.H.S.}=2xe^x+x^2e^x$

Thus, the statement is true when $n=1$. R1

Assume that the statement is true when $n=k$. M1

$f^{(k)}(x)=(k(k-1)+2kx+x^2)e^x$

When $n=k+1$,

$f^{(k+1)}(x)=\frac{\mathrm{d}}{\mathrm{d}x}(f^{(k)}(x))$ M1

$f^{(k+1)}(x)=(0+2k+2x)(e^x)+(k(k-1)+2kx+x^2)(e^x)$ A1

$f^{(k+1)}(x)=(2k+2x+k^2-k+2kx+x^2)e^x$ A1

$f^{(k+1)}(x)=(k^2+k+2kx+2x+x^2)e^x$

$f^{(k+1)}(x)=((k+1)k+2(k+1)x+x^2)e^x$ A1

Thus, the statement is true when $n=k+1$.

Therefore, the statement is true for all $n\in\mathbb{Z}^+$. R1

[7]

3. When $n=2$,

$$\text{L.H.S.}=\frac{\mathrm{d}}{\mathrm{d}x}((1)(7^x)+(x)(7^x\ln 7))$$

$$\text{L.H.S.}=\frac{\mathrm{d}}{\mathrm{d}x}(7^x+x7^x\ln 7)$$

$$\text{L.H.S.}=7^x\ln 7+(1)(7^x\ln 7)+(x)(7^x(\ln 7)^2)$$

$$\text{L.H.S.}=2\cdot 7^x\ln 7+x7^x(\ln 7)^2$$

$$\text{R.H.S.}=2\cdot 7^x(\ln 7)^{2-1}+x7^x(\ln 7)^2$$

$$\text{R.H.S.}=2\cdot 7^x\ln 7+x7^x(\ln 7)^2$$

Thus, the statement is true when $n=2$. R1

Assume that the statement is true when $n=k$. M1

$$f^{(k)}(x)=k7^x(\ln 7)^{k-1}+x7^x(\ln 7)^k$$

When $n=k+1$,

$$f^{(k+1)}(x)=\frac{\mathrm{d}}{\mathrm{d}x}(f^{(k)}(x))$$ M1

$$f^{(k+1)}(x)=k7^x(\ln 7)^k+(1)(7^x(\ln 7)^k)+(x)(7^x(\ln 7)^{k+1})$$ A1

$$f^{(k+1)}(x)=k7^x(\ln 7)^k+7^x(\ln 7)^k+x7^x(\ln 7)^{k+1}$$ A1

$$f^{(k+1)}(x)=(k+1)7^x(\ln 7)^k+x7^x(\ln 7)^{k+1}$$

$$f^{(k+1)}(x)=(k+1)7^x(\ln 7)^{k+1-1}+x7^x(\ln 7)^{k+1}$$ A1

Thus, the statement is true when $n=k+1$.

Therefore, the statement is true for all $n\in\mathbb{Z}^+$, $n\geq 2$. R1

[7]

4. When $n=1$,

$$\text{L.H.S.}=-1\left(-\frac{1}{2}\right)(1+x)^{-\frac{3}{2}}(1)$$

$$\text{L.H.S.}=\frac{1}{2(1+x)^{\frac{3}{2}}}$$

$$\text{R.H.S.}=\frac{(-1)^{1+1}(2(1))!}{2^{2(1)}1!}(1+x)^{-1-\frac{1}{2}}$$

$$\text{R.H.S.}=\frac{1}{2(1+x)^{\frac{3}{2}}}$$

Thus, the statement is true when $n=1$. R1

Assume that the statement is true when $n=k$. M1

$$f^{(k)}(x)=\frac{(-1)^{k+1}(2k)!}{2^{2k}k!}(1+x)^{-k-\frac{1}{2}}$$

When $n=k+1$,

$$f^{(k+1)}(x)=\frac{\mathrm{d}}{\mathrm{d}x}(f^{(k)}(x))$$ M1

$$f^{(k+1)}(x)=\frac{(-1)^{k+1}(2k)!}{2^{2k}k!}\left(-k-\frac{1}{2}\right)(1+x)^{-k-\frac{3}{2}}(1)$$ A1

$$f^{(k+1)}(x)=\frac{(-1)^{k+1}(2k)!}{2^{2k}k!}\left(-\frac{2k+1}{2}\right)(1+x)^{-k-\frac{3}{2}}$$ A1

$$f^{(k+1)}(x)=\frac{(-1)^{k+2}(2k+1)(2k)!}{2^{2k+1}k!}(1+x)^{-k-\frac{3}{2}}$$

$$f^{(k+1)}(x)=\frac{(-1)^{k+2}(2k+2)(2k+1)(2k)!}{2^{2k+1}k!(2)(k+1)}(1+x)^{-k-\frac{3}{2}}$$ A1

$$f^{(k+1)}(x)=\frac{(-1)^{k+2}(2k+2)!}{2^{2k+2}(k+1)!}(1+x)^{-k-\frac{3}{2}}$$

$$f^{(k+1)}(x)=\frac{(-1)^{k+1+1}(2(k+1))!}{2^{2(k+1)}(k+1)!}(1+x)^{-(k+1)-\frac{1}{2}}$$ A1

Thus, the statement is true when $n=k+1$.

Therefore, the statement is true for all $n\in\mathbb{Z}^{+}$. R1

[8]

Exercise 43

1. $x+y-e^{2y}\ln x=2$

$\frac{d}{dx}(x)+\frac{d}{dx}(y)-\frac{d}{dx}(e^{2y}\ln x)=\frac{d}{dx}(2)$ (M1) for valid approach

$1+\frac{dy}{dx}-\left(\left(2e^{2y}\frac{dy}{dx}\right)(\ln x)+(e^{2y})\left(\frac{1}{x}\right)\right)=0$ (A2) for correct approach

$1+\frac{dy}{dx}-2e^{2y}\ln x\frac{dy}{dx}-\frac{e^{2y}}{x}=0$

$\frac{dy}{dx}-2e^{2y}\ln x\frac{dy}{dx}=\frac{e^{2y}}{x}-1$

$(1-2e^{2y}\ln x)\frac{dy}{dx}=\frac{e^{2y}-x}{x}$

$\frac{dy}{dx}=\frac{e^{2y}-x}{x(1-2e^{2y}\ln x)}$ A1

The required gradient

$=\frac{e^{2(1)}-1}{1(1-2e^{2(1)}\ln 1)}$ (A1) for substitution

$=\frac{e^{2}-1}{1(1-0)}$

$=e^{2}-1$ A1

[6]

2. $(x-3)^2+4y^2=4(4y-3)$

$\frac{d}{dx}(x-3)^2+\frac{d}{dx}(4y^2)=\frac{d}{dx}(16y-12)$ (M1) for valid approach

$2(x-3)(1)+8y\frac{dy}{dx}=16\frac{dy}{dx}-0$ (A1) for correct approach

$2x-6+8y\frac{dy}{dx}=16\frac{dy}{dx}$

$x-3=8\frac{dy}{dx}-4y\frac{dy}{dx}$

$x-3=(8-4y)\frac{dy}{dx}$

$\frac{dy}{dx}=\frac{x-3}{8-4y}$ A1

$\therefore \frac{x-3}{8-4y}=0$ (A1) for substitution

$x-3=0$

$x=3$

$\therefore (3-3)^2+4y^2=4(4y-3)$ (M1) for substitution

$4y^2=4(4y-3)$

$y^2=4y-3$

$y^2-4y+3=0$ A1

$(y-1)(y-3)=0$

$y=1$ or $y=3$

Thus, the required coordinates are $(3,1)$ and $(3,3)$. A2

[8]

3. $xy + y^2 = 3$

$$\frac{\mathrm{d}}{\mathrm{d}x}(xy) + \frac{\mathrm{d}}{\mathrm{d}x}(y^2) = \frac{\mathrm{d}}{\mathrm{d}x}(3) \qquad \text{M1}$$

$$(1)(y) + (x)\left(\frac{\mathrm{d}y}{\mathrm{d}x}\right) + 2y\frac{\mathrm{d}y}{\mathrm{d}x} = 0 \qquad \text{A2}$$

$$y + x\frac{\mathrm{d}y}{\mathrm{d}x} + 2y\frac{\mathrm{d}y}{\mathrm{d}x} = 0$$

$$(x+2y)\frac{\mathrm{d}y}{\mathrm{d}x} = -y$$

$$\frac{\mathrm{d}y}{\mathrm{d}x} = -\frac{y}{x+2y} \qquad \text{A1}$$

$$\frac{\mathrm{d}^2y}{\mathrm{d}x^2} = -\frac{(x+2y)\left(\frac{\mathrm{d}y}{\mathrm{d}x}\right) - (y)\left(1 + 2\frac{\mathrm{d}y}{\mathrm{d}x}\right)}{(x+2y)^2} \qquad \text{A1}$$

$$\frac{\mathrm{d}^2y}{\mathrm{d}x^2} = -\frac{x\frac{\mathrm{d}y}{\mathrm{d}x} - y}{(x+2y)^2}$$

$$\frac{\mathrm{d}^2y}{\mathrm{d}x^2} = \frac{y - x\frac{\mathrm{d}y}{\mathrm{d}x}}{(x+2y)^2} \qquad \text{A1}$$

$$\therefore (x+2y)\frac{\mathrm{d}^2y}{\mathrm{d}x^2} + \frac{\mathrm{d}y}{\mathrm{d}x} - \frac{xy}{(x+2y)^2}$$

$$= (x+2y)\left(\frac{y - x\frac{\mathrm{d}y}{\mathrm{d}x}}{(x+2y)^2}\right) + \frac{\mathrm{d}y}{\mathrm{d}x} - \frac{xy}{(x+2y)^2} \qquad \text{M1}$$

$$= (x+2y)\left(\frac{y - x\left(-\frac{y}{x+2y}\right)}{(x+2y)^2}\right) - \frac{y}{x+2y} - \frac{xy}{(x+2y)^2}$$

$$= \frac{y}{x+2y} + \frac{xy}{(x+2y)^2} - \frac{y}{x+2y} - \frac{xy}{(x+2y)^2}$$

$$= 0 \qquad \text{AG}$$

[7]

4. $e^x + e^y + 4x = 0$

$\frac{\mathrm{d}}{\mathrm{d}x}(e^x) + \frac{\mathrm{d}}{\mathrm{d}x}(e^y) + \frac{\mathrm{d}}{\mathrm{d}x}(4x) = 0$ M1

$e^x + e^y \frac{\mathrm{d}y}{\mathrm{d}x} + 4 = 0$ A1

$e^y \frac{\mathrm{d}y}{\mathrm{d}x} = -(e^x + 4)$

$\frac{\mathrm{d}y}{\mathrm{d}x} = -\frac{e^x + 4}{e^y}$ A1

$\frac{\mathrm{d}^2 y}{\mathrm{d}x^2} = -\frac{(e^y)(e^x + 0) - (e^x + 4)\left(e^y \frac{\mathrm{d}y}{\mathrm{d}x}\right)}{(e^y)^2}$ A1

$\frac{\mathrm{d}^2 y}{\mathrm{d}x^2} = -\frac{e^{x+y} - (e^{x+y} + 4e^y)\frac{\mathrm{d}y}{\mathrm{d}x}}{e^{2y}}$

$\frac{\mathrm{d}^2 y}{\mathrm{d}x^2} = (e^{x-y} + 4e^{-y})\frac{\mathrm{d}y}{\mathrm{d}x} - e^{x-y}$ A1

$\therefore e^{2y}\frac{\mathrm{d}^2 y}{\mathrm{d}x^2} + (4 + e^x)^2 + e^{x+y}$

$= e^{2y}\left((e^{x-y} + 4e^{-y})\frac{\mathrm{d}y}{\mathrm{d}x} - e^{x-y}\right) + (4 + e^x)^2 + e^{x+y}$ M1

$= \left((e^{x+y} + 4e^y)\left(-\frac{e^x + 4}{e^y}\right) - e^{x+y}\right) + (4 + e^x)^2 + e^{x+y}$

$= -(e^x + 4)^2 - e^{x+y} + (4 + e^x)^2 + e^{x+y}$

$= 0$ AG

[6]

Exercise 44

1. (a) $e^x y = 12 - y^2$

$\frac{d}{dx}(e^x y) = \frac{d}{dx}(12) - \frac{d}{dx}(y^2)$ M1

$(e^x)(y) + (e^x)\left(\frac{dy}{dx}\right) = 0 - 2y\frac{dy}{dx}$ A2

$e^x y = -e^x \frac{dy}{dx} - 2y\frac{dy}{dx}$ M1

$e^x y = -(e^x + 2y)\frac{dy}{dx}$

$\frac{dy}{dx} = -\frac{e^x y}{e^x + 2y}$ AG

[4]

(b) (i) $3x + 7y - 21 = 0$

$7y = 21 - 3x$

$y = 3 - \frac{3}{7}x$ (M1) for valid approach

$\therefore e^x\left(3 - \frac{3}{7}x\right) = 12 - \left(3 - \frac{3}{7}x\right)^2$ (M1) for substitution

$\left(3 - \frac{3}{7}x\right)^2 + e^x\left(3 - \frac{3}{7}x\right) - 12 = 0$

By considering the graph of

$y = \left(3 - \frac{3}{7}x\right)^2 + e^x\left(3 - \frac{3}{7}x\right) - 12$,

$x = 0$ or $x = 6.9737895$ (*Rejected*). A1

$\therefore y = 3 - \frac{3}{7}(0)$

$y = 3$

Thus, the coordinates of P are $(0, 3)$. A1

(ii) The gradient of the tangent at P

$= -\frac{e^0(3)}{e^0 + 2(3)}$ (M1) for substitution

$= -\frac{3}{7}$ A1

[6]

(c) $e^0 y = 12 - y^2$

$y^2 + y - 12 = 0$ (A1) for correct equation

$(y+4)(y-3) = 0$

$y = -4$ or $y = 3$ (*Rejected*) A1

The gradient of the normal at Q

$= -1 \div -\dfrac{e^0(-4)}{e^0 + 2(-4)}$ (M1) for substitution

$= -1 \div -\dfrac{4}{7}$

$= \dfrac{7}{4}$ A1

[4]

2. (a) $2x^2 + axy + y^2 = 56$

$\frac{d}{dx}(2x^2) + \frac{d}{dx}(axy) + \frac{d}{dx}(y^2) = \frac{d}{dx}(56)$ M1

$4x + (a)(y) + (ax)\left(\frac{dy}{dx}\right) + 2y\frac{dy}{dx} = 0$ A2

$ax\frac{dy}{dx} + 2y\frac{dy}{dx} = -(4x + ay)$ M1

$(ax + 2y)\frac{dy}{dx} = -(4x + ay)$

$\frac{dy}{dx} = -\frac{4x + ay}{ax + 2y}$ AG

[4]

(b) (i) -1 A1

(ii) The gradient of the tangent at P

$= -1 \div -1$

$= 1$ (A1) for correct value

$\therefore -\frac{4(2) + a(-6)}{2a + 2(-6)} = 1$ (M1) for setting equation

$-\frac{8 - 6a}{2a - 12} = 1$

$6a - 8 = 2a - 12$ (M1) for valid approach

$4a = -4$

$a = -1$ A1

[5]

(c) $\frac{dy}{dx} = 0$

$-\frac{4x - y}{-x + 2y} = 0$ A1

$4x - y = 0 \Rightarrow y = 4x$ A1

$\therefore 2x^2 - x(4x) + (4x)^2 = 56$ M1

$2x^2 - 4x^2 + 16x^2 = 56$

$14x^2 = 56$

$x^2 = 4$

$x = -2$ or $x = 2$ A1

When $x = -2$, $y = 4(-2) = -8$. M1

When $x = 2$, $y = 4(2) = 8$.

Thus, that the coordinates of Q and R are $(2, 8)$ and $(-2, -8)$. AG

[5]

3. (a) $5x^2 - 2xy - 3y^2 - 16x = 0$

$\frac{d}{dx}(5x^2) - \frac{d}{dx}(2xy) - \frac{d}{dx}(3y^2) - \frac{d}{dx}(16x) = \frac{d}{dx}(0)$ M1

$10x - \left((2)(y) + (2x)\left(\frac{dy}{dx}\right)\right) - 6y\frac{dy}{dx} - 16 = 0$ A2

$10x - 2y - 2x\frac{dy}{dx} - 6y\frac{dy}{dx} - 16 = 0$

$5x - y - x\frac{dy}{dx} - 3y\frac{dy}{dx} - 8 = 0$

$x\frac{dy}{dx} + 3y\frac{dy}{dx} = 5x - y - 8$ M1

$(x + 3y)\frac{dy}{dx} = 5x - y - 8$

$\frac{dy}{dx} = \frac{5x - y - 8}{x + 3y}$ AG

[4]

(b) The gradient of the tangent

$= -1 \div -\frac{1}{5}$

$= 5$ (A1) for correct value

$\therefore \frac{5x - y - 8}{x + 3y} = 5$ (M1) for setting equation

$5x - y - 8 = 5(x + 3y)$

$5x - y - 8 = 5x + 15y$

$16y = -8$

$y = -0.5$

$\therefore 5x^2 - 2x(-0.5) - 3(-0.5)^2 - 16x = 0$ (M1) for substitution

$5x^2 + x - 0.75 - 16x = 0$

$5x^2 - 15x - 0.75 = 0$

$20x^2 - 60x - 3 = 0$ A1

By considering the graph of $y = 20x^2 - 60x - 3$,

$x = -0.049193$ or $x = 3.0491933$.

Thus, that the coordinates of Q and R are

$(-0.0492, -0.5)$ and $(3.05, -0.5)$. A2

[6]

$$(x+3y)\frac{d}{dx}(5x-y-8)$$

(c) $$\frac{d^2y}{dx^2}=\frac{-(5x-y-8)\frac{d}{dx}(x+3y)}{(x+3y)^2}$$ (M1) for valid approach

$$\frac{d^2y}{dx^2}=\frac{(x+3y)\left(5-\frac{dy}{dx}-0\right)-(5x-y-8)\left(1+3\frac{dy}{dx}\right)}{(x+3y)^2}$$ (A1) for correct approach

$$5x+15y-(x+3y)\frac{dy}{dx}$$

$$\frac{d^2y}{dx^2}=\frac{-(5x-y-8)-(15x-3y-24)\frac{dy}{dx}}{(x+3y)^2}$$

$$\frac{d^2y}{dx^2}=\frac{16y+8+(-16x+24)\frac{dy}{dx}}{(x+3y)^2}$$ A1

$$\therefore (x+3y)^2\left(\frac{16y+8+(-16x+24)\frac{dy}{dx}}{(x+3y)^2}\right)$$ M1

$$\geq 8(3-2x)\frac{dy}{dx}$$

$$16y+8+(-16x+24)\frac{dy}{dx}\geq(24-16x)\frac{dy}{dx}$$ (M1) for valid approach

$$16y\geq -8$$

$$y\geq -\frac{1}{2}$$ A1

[6]

4. (a) $x + xy = 8 + y^2$

$\frac{d}{dx}(x) + \frac{d}{dx}(xy) = \frac{d}{dx}(8) + \frac{d}{dx}(y^2)$ M1

$1 + \left((1)(y) + (x)\left(\frac{dy}{dx}\right)\right) = 0 + 2y\frac{dy}{dx}$ A2

$1 + y + x\frac{dy}{dx} = 2y\frac{dy}{dx}$

$2y\frac{dy}{dx} - x\frac{dy}{dx} = 1 + y$ M1

$(2y - x)\frac{dy}{dx} = 1 + y$

$\frac{dy}{dx} = \frac{1+y}{2y-x}$ AG

[4]

(b) The gradient of the normal

$= -1 \div \frac{dy}{dx}$

$= -1 \div \frac{1+y}{2y-x}$

$= \frac{x-2y}{1+y}$ (A1) for correct approach

$\therefore \frac{x-2y}{1+y} = 0$ (M1) for setting equation

$x - 2y = 0$

$x = 2y$

$\therefore 2y + (2y)y = 8 + y^2$ (M1) for substitution

$2y + 2y^2 = 8 + y^2$

$y^2 + 2y - 8 = 0$ A1

$(y+4)(y-2) = 0$

$y = -4$ or $y = 2$ (A1) for correct values

When $y = -4$, $x = 2(-4) = -8$.

When $y = 2$, $x = 2(2) = 4$.

Thus, that the coordinates of P and Q are

$(-8, -4)$ and $(4, 2)$. A2

[7]

(c) $$\frac{d^2y}{dx^2}=\frac{(2y-x)\frac{d}{dx}(1+y)-(1+y)\frac{d}{dx}(2y-x)}{(2y-x)^2}$$ M1

$$\frac{d^2y}{dx^2}=\frac{(2y-x)\left(0+\frac{dy}{dx}\right)-(1+y)\left(2\frac{dy}{dx}-1\right)}{(2y-x)^2}$$ A1

$$\frac{d^2y}{dx^2}=\frac{(2y-x)\frac{dy}{dx}-(2+2y)\frac{dy}{dx}+1+y}{(2y-x)^2}$$ M1

$$\frac{d^2y}{dx^2}=\frac{(-2-x)\frac{dy}{dx}+1+y}{(2y-x)^2}$$ A1

$$\therefore \frac{(-2-x)\frac{dy}{dx}+1+y}{(2y-x)^2}-\frac{1}{1+y}\left(\frac{dy}{dx}\right)^2+\frac{(x+2)(y+1)}{(2y-x)^3}$$ M1

$$=\frac{(-2-x)\left(\frac{1+y}{2y-x}\right)+1+y}{(2y-x)^2}-\frac{1}{1+y}\left(\frac{1+y}{2y-x}\right)^2$$ M1

$$+\frac{(x+2)(y+1)}{(2y-x)^3}$$

$$=-\frac{(2+x)(1+y)}{(2y-x)^3}+\frac{1+y}{(2y-x)^2}$$

$$-\frac{1+y}{(2y-x)^2}+\frac{(x+2)(y+1)}{(2y-x)^3}$$

$$=0$$ AG

[6]