AA HL Practice Set 4 Paper 2 Solution

Section A

1. (a) (i) $(3, -127)$ A2

(ii) $f(x) = 3(x-3)^2 - 127$ A2

[4]

(b) $3x^2 - 18x - 100 = -52$

$3x^2 - 18x - 48 = 0$ (A1) for correct equation

$3(x+2)(x-8) = 0$

$x = -2$ or $x = 8$ A2

[3]

2. (a) p is negative as the first turning point is a minimum point. R1

$p = -\frac{4.3}{2}$ A1

$p = -2.15$ AG

[2]

(c) (i) The period

$= 13.75 - 2.75$ (M1) for valid approach

$= 11$ hours (A1) for correct value

$\therefore q = \frac{2\pi}{11}$ A1

(ii) $r = \frac{(1.9 + 4.3) + 1.9}{2}$ (M1) for valid approach

$r = 4.05$ A1

[5]

3. (a) $B\hat{A}C$

$= \pi - 0.88 - 1.23$ (M1) for valid approach

$= 1.031592654$ A1

$\frac{AB}{\sin A\hat{C}B} = \frac{BC}{\sin B\hat{A}C}$ (M1) for sine rule

$\frac{AB}{\sin 1.23} = \frac{20}{\sin 1.031592654}$ (A1) for substitution

$AB = 21.96641928$ cm

$AB = 22.0$ cm A1

[5]

(b) $AB^2 = OA^2 + OB^2 - 2(OA)(OB)\cos A\hat{O}B$ M1

$AB^2 = r^2 + r^2 - 2(r)(r)\cos A\hat{O}B$ A1

$AB^2 = 2r^2 - 2r^2 \cos A\hat{O}B$

$AB^2 = 2r^2(1 - \cos A\hat{O}B)$ A1

$r^2 = \frac{AB^2}{2(1 - \cos A\hat{O}B)}$ AG

[3]

4. (a) The common ratio r

$= \frac{3k^2 - 4k^3}{k^2}$ (M1) for valid approach

$= 3 - 4k$ A1

[2]

(b) S_∞ exists if $-1 < r < 1$. R1

$\therefore -1 < 3 - 4k < 1$ M1

$-1 < 4k - 3 < 1$

$2 < 4k < 4$ A1

$\frac{1}{2} < k < 1$ AG

[3]

5. The general term

$$=\binom{9}{r}\left(\frac{x}{h^2}\right)^{9-r}\left(-\frac{h}{x^2}\right)^r$$ (M1) for valid expansion

$$=\binom{9}{r}(-1)^r h^{3r-18}x^{9-3r}$$

$9-3r=0$ (A1) for correct equation

$3r=9$

$r=3$ (A1) for correct value

The required term

$$=\binom{9}{3}(-1)^3 h^{3(3)-18}x^{9-3(3)}$$

$$=-\frac{84}{h^9}$$ (A1) for correct term

$$-\frac{84}{h^9}=-\frac{21}{65536}$$ (M1) for setting equation

$h^9=262144$

$h=4$ A1

[6]

6. (a) Let $\frac{x^2+9}{(4-x)(5-2x)} \equiv A+\frac{B}{4-x}+\frac{C}{5-2x}$, where A, B and C are constants.

$\frac{x^2+9}{(4-x)(5-2x)} \equiv \frac{A(4-x)(5-2x)}{(4-x)(5-2x)}$ M1

$+\frac{B(5-2x)}{(4-x)(5-2x)}+\frac{C(4-x)}{(4-x)(5-2x)}$

$\frac{x^2+9}{(4-x)(5-2x)}$

$\equiv \frac{20A-13Ax+2Ax^2+5B-2Bx+4C-Cx}{(4-x)(5-2x)}$

$x^2+9 \equiv 2Ax^2+(-13A-2B-C)x+(20A+5B+4C)$ A1

$2A=1$

$A=\frac{1}{2}$ A1

$0=-13\left(\frac{1}{2}\right)-2B-C$

$C=-\frac{13}{2}-2B$

$9=20A+5B+4C$

$\therefore 9=20\left(\frac{1}{2}\right)+5B+4\left(-\frac{13}{2}-2B\right)$ A1

$9=10+5B-26-8B$

$25=-3B$

$B=-\frac{25}{3}$ A1

$\therefore C=-\frac{13}{2}-2\left(-\frac{25}{3}\right)$

$C=\frac{61}{6}$ A1

[6]

(b) $g(x)=-\frac{(4-x)(5-2x)}{x^2+9}$

The discriminant of x^2+9

$=0^2-4(1)(9)$ A1

$=-36$

<0

Therefore, the denominator is always nonzero.

Thus, $g(x)$ has no vertical asymptote. AG

[1]

7. (a) $\left\{x : -5 \le x \le \frac{2}{3}\right\}$ A2

[2]

(b) $f(x) = (3x-2)^2$

$y = (3x-2)^2$

$\Rightarrow x = (3y-2)^2$ (M1) for swapping variables

$-\sqrt{x} = 3y - 2$

$-\sqrt{x} + 2 = 3y$

$y = \dfrac{-\sqrt{x}+2}{3}$

$\therefore f^{-1}(x) = \dfrac{-\sqrt{x}+2}{3}$ A1

[2]

(c) $(f \circ g)(x) = x^4$

$g(x) = f^{-1}(x^4)$ M1

$g(x) = \dfrac{-\sqrt{x^4}+2}{3}$

$g(x) = \dfrac{-x^2+2}{3}$ A1

[2]

8. $\binom{12}{2} \times \binom{10}{r} \times \binom{10-r}{10-r} = 7920$ M1A1

$\binom{10}{r} = 120$ (A1) for simplification

$\binom{10}{r} = \binom{10}{3}$ or $\binom{10}{r} = \binom{10}{7}$

$r = 3$ or $r = 7$ A2

[5]

9. (a) The standard deviation of X

$$=\sqrt{E(X^2)-(E(X))^2}$$ (M1) for valid approach

$$=\sqrt{\int_{-4}^{0}x^2\cdot\left(\frac{1}{20}x+\frac{1}{5}\right)dx+\int_{0}^{3}x^2\cdot\left(-\frac{1}{15}x^2+\frac{2}{15}x+\frac{1}{5}\right)dx-\left(\frac{13}{60}\right)^2}$$ A1

$$=\sqrt{2.279722222}$$

$$=1.509874903$$

$$=1.51$$ A1

[3]

(b) $P(|X|>2)$

$$=P(X>2\text{ or }X<-2)$$ (M1) for valid approach

$$=P(X<-2)+P(X>2)$$

$$=\int_{-4}^{-2}\left(\frac{1}{20}x+\frac{1}{5}\right)dx+\int_{2}^{3}\left(-\frac{1}{15}x^2+\frac{2}{15}x+\frac{1}{5}\right)dx$$ A1

$$=\frac{19}{90}$$ A1

[3]

Section B

10. (a) (i) $a_1(t) = \frac{20-30}{2-0}$ M1A1

$a_1(t) = -5$ AG

(ii) $v_1(t) = -5t + 30$ A2

[4]

(b) The total distance the marble travelled

$= \int_0^2 |v_1(t)|\, dt$ (M1) for valid approach

$= \int_0^2 |-5t+30|\, dt$ (A1) for correct formula

$= 50 \text{ cm}$ A1

[3]

(c) (i) $v_2(2) = 20$

$\therefore 20e^{b-0.2(2)} = 20$ M1

$e^{b-0.4} = 1$

$b - 0.4 = 0$ A1

$b = 0.4$ AG

(ii) $\int_2^c |v_2(t)|\, dt = 50$

$\int_2^c 20e^{0.4-0.2t}\, dt = 50$ (M1) for setting equation

Let $u = 0.4 - 0.2t$
$\frac{du}{dt} = -0.2 \Rightarrow -100du = 20dt$
$t = c \Rightarrow u = 0.4 - 0.2c$
$t = 2 \Rightarrow u = 0.4 - 0.2(2) = 0$

(A1) for substitution

$\int_0^{0.4-0.2c} -100e^u\, du = 50$ A1

$\left[-100e^u\right]_0^{0.4-0.2c} = 50$

$e^{0.4-0.2c} - e^0 = -0.5$ (M1) for substitution

$e^{0.4-0.2c} = 0.5$

$0.4 - 0.2c = \ln 0.5$

$0.4 - \ln 0.5 = 0.2c$

$c = 5.465735903$

$c = 5.47$ A1

[7]

11. (a) The coordinates of A, B' and C are $(-3, 0, 0)$, $(0, 4, 0)$ and $(0, 0, 8)$ respectively. A1

$\mathbf{n} = \overrightarrow{\text{AB}'} \times \overrightarrow{\text{AC}}$ M1

$\mathbf{n} = (3\mathbf{i} + 4\mathbf{j}) \times (3\mathbf{i} + 8\mathbf{k})$ A1

$$\mathbf{n} = \begin{pmatrix} (4)(8)-(0)(0) \\ (0)(3)-(3)(8) \\ (3)(0)-(4)(3) \end{pmatrix}$$

$\mathbf{n} = 32\mathbf{i} - 24\mathbf{j} - 12\mathbf{k}$ A1

The Cartesian equation of the plane π_2:

$(x\mathbf{i} + y\mathbf{j} + z\mathbf{k}) \cdot (32\mathbf{i} - 24\mathbf{j} - 12\mathbf{k})$

$= -3\mathbf{i} \cdot (32\mathbf{i} - 24\mathbf{j} - 12\mathbf{k})$ M1A1

$32x - 24y - 12z = (-3)(32) + (0)(-24) + (0)(-12)$

$32x - 24y - 12z = -96$

$8x - 6y - 3z = -24$ AG

[6]

(b) The coordinates of B are $(0, -4, 0)$. (A1) for correct values

The volume of the pyramid ABCC'

$= \frac{1}{3}\left(\frac{(\text{BB}')(\text{OA})}{2}\right)(\text{OC})$ (M1) for valid approach

$= \frac{1}{3}\left(\frac{(4-(-4))(0-(-3))}{2}\right)(8)$ A1

$= 32$ A1

[4]

(c) $\mathbf{n}_1 = 8\mathbf{i}+6\mathbf{j}-3\mathbf{k}$

$\mathbf{n}_2 = 8\mathbf{i}-6\mathbf{j}-3\mathbf{k}$ (A1) for correct values

Let θ be the obtuse angle between the planes.

$\mathbf{n}_1 \cdot \mathbf{n}_2 = |\mathbf{n}_1||\mathbf{n}_2|\cos\theta$ (M1) for valid approach

$(8)(8)+(6)(-6)+(-3)(-3)$

$=(\sqrt{8^2+6^2+(-3)^2})(\sqrt{8^2+(-6)^2+(-3)^2})\cos\theta$ (A1) for substitution

$37=(\sqrt{109})(\sqrt{109})\cos\theta$

$\cos\theta=\frac{37}{109}$ A1

$\theta = 70.15665929^\circ$

The required obtuse angle

$=180^\circ - 70.15665929^\circ$

$=109.8433407^\circ$

$=110^\circ$ A1

[5]

(d) The mid-point of BC

$=\left(\frac{0+0}{2}, \frac{-4+0}{2}, \frac{0+8}{2}\right)$

$=(0,-2,4)$ (A1) for correct values

$\mathbf{n}_3 = \mathbf{n}_1 \times \mathbf{n}_2$ (M1) for valid approach

$\mathbf{n}_3 = (8\mathbf{i}+6\mathbf{j}-3\mathbf{k})\times(8\mathbf{i}-6\mathbf{j}-3\mathbf{k})$

$$\mathbf{n}_3 = \begin{pmatrix} (6)(-3)-(-3)(-6) \\ (-3)(8)-(8)(-3) \\ (8)(-6)-(6)(8) \end{pmatrix}$$

$\mathbf{n}_3 = -36\mathbf{i}-96\mathbf{k}$ A1

The vector equation of the line:

$$\mathbf{r} = \begin{pmatrix} 0 \\ -2 \\ 4 \end{pmatrix} + t\begin{pmatrix} -36 \\ 0 \\ -96 \end{pmatrix}$$ A1

$$\begin{cases} x=-36t \\ y=-2 \\ z=4-96t \end{cases}$$

$\frac{x}{-36}=\frac{z-4}{-96}, y=-2$ A1

[5]

12. (a) (i) $x^2\frac{\mathrm{d}y}{\mathrm{d}x}+6y=x^3e^{x^2+\frac{6}{x}}$

$\frac{\mathrm{d}y}{\mathrm{d}x}+\frac{6}{x^2}y=xe^{x^2+\frac{6}{x}}$ (A1) for correct approach

The integrating factor

$=e^{\int\frac{6}{x^2}\mathrm{d}x}$ (M1) for valid approach

$=e^{-\frac{6}{x}}$ A1

$\therefore e^{-\frac{6}{x}}\frac{\mathrm{d}y}{\mathrm{d}x}+e^{-\frac{6}{x}}\cdot\frac{6}{x^2}y=e^{-\frac{6}{x}}\cdot xe^{x^2+\frac{6}{x}}$ (M1) for valid approach

$e^{-\frac{6}{x}}\frac{\mathrm{d}y}{\mathrm{d}x}+e^{-\frac{6}{x}}\cdot\frac{6}{x^2}y=xe^{x^2}$

$\frac{\mathrm{d}}{\mathrm{d}x}\left(ye^{-\frac{6}{x}}\right)=xe^{x^2}$ (A1) for correct approach

$ye^{-\frac{6}{x}}=\int xe^{x^2}\mathrm{d}x$

Let $u=x^2$. (M1) for substitution

$\frac{\mathrm{d}u}{\mathrm{d}x}=2x\Rightarrow\frac{1}{2}\mathrm{d}u=x\mathrm{d}x$

$\therefore ye^{-\frac{6}{x}}=\int e^u\cdot\frac{1}{2}\mathrm{d}u$ (A1) for correct working

$ye^{-\frac{6}{x}}=\frac{1}{2}\int e^u\mathrm{d}u$

$ye^{-\frac{6}{x}}=\frac{1}{2}e^u+C$

$ye^{-\frac{6}{x}}=\frac{1}{2}e^{x^2}+C$ A1

$ye^{-\frac{6}{x}}=\frac{e^{x^2}+C}{2}$

$y=\frac{e^{\frac{6}{x}}(e^{x^2}+C)}{2}$ A1

$\frac{e^7}{2}=\frac{e^{\frac{6}{1}}(e^{1^2}+C)}{2}$ (M1) for substitution

$\frac{e^7}{2}=\frac{e^7+Ce^6}{2}$

$Ce^6=0$

$C=0$

$$\therefore y=\frac{e^{\frac{6}{x}+x^2}}{2}$$ A1

(ii) $\frac{e^{11}}{2}$ A1

[12]

(b) (i) $\begin{cases} x_{n+1}=x_n+0.1 \\ y_{n+1}=y_n+0.1\left.\frac{\mathrm{d}y}{\mathrm{d}x}\right|_{(x_n,\,y_n)} \end{cases}$ M1

$x_0=1,\ y_0=\frac{e^7}{2}$ A1

$x_1=1+0.1$

$x_1=1.1$

$y_1=\frac{e^7}{2}+0.1\left(1e^{1^2+\frac{6}{1}}-\frac{6}{1^2}\left(\frac{e^7}{2}\right)\right)$ M1A1

$y_1=\frac{e^7}{2}-0.2e^7$

$y_1=\frac{3e^7}{10}$ AG

(ii) 23435.5461 A2

[6]

(c) $23435.5461<\frac{e^{11}}{2}$ R1

[1]