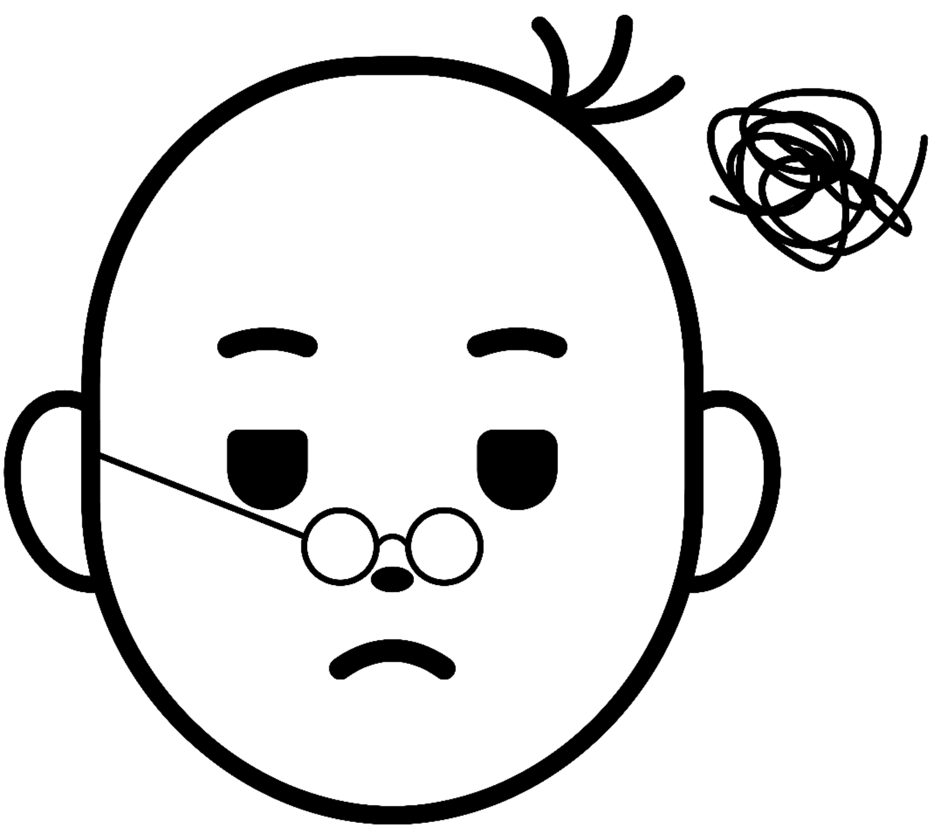
Your Intensive Notes Analysis and Approaches Higher Level for IBDP Mathematics



Functions

Topics Covered

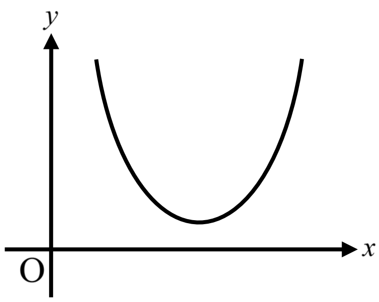
1 Quadratic Functions

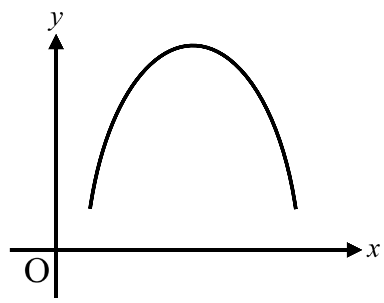
Important Notes

Quadratic function: A polynomial function with the greatest power of x equals to 2

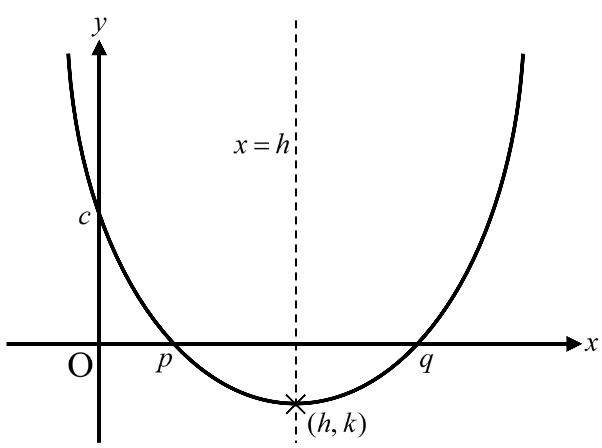
Properties of a quadratic function in its general form $y = ax^2 + bx + c$, $a \neq 0$

1. $a > 0$: Opens upward $\quad$ $a < 0$: Opens downward





2. c: y-intercept of the graph
3. $y = a(x-p)(x-q)$: Factored form with x-intercept(s) p and q, $p \leq q$
4. $y = a(x-h)^2 + k$: Vertex form with coordinates of the vertex (h, k)
5. $x = h$: Equation of the axis of symmetry of the graph
6. $h = -\frac{b}{2a} = \frac{p+q}{2}$: x-coordinate of the vertex of the graph
7. $k = ah^2 + bh + c$: y-coordinate of the vertex of the graph, which is also the extreme (maximum when $a < 0$/minimum when $a > 0$) value of y



Methods of solving a quadratic equation $ax^2+bx+c=0$, $a \neq 0$:

1. Factorization by cross method
2. $x=\dfrac{-b \pm \sqrt{b^2-4ac}}{2a}$: Quadratic Formula
3. Method of completing the square

Root(s) of the quadratic equation $ax^2+bx+c=0$: x-intercept(s) of the graph of the corresponding quadratic function $y=ax^2+bx+c$

The discriminant $\Delta=b^2-4ac$ of the quadratic equation $ax^2+bx+c=0$:

1. $\Delta > 0$: The quadratic equation has two distinct real roots
2. $\Delta = 0$: The quadratic equation has two equal real roots (one double real root)
3. $\Delta < 0$: The quadratic equation has no real root (two complex roots)

Example 2.1

A quadratic function is defined as $y = x^2 - 2x - 8$, $x \in \mathbb{R}$.

(a) Write down the y-intercept of the quadratic graph.

[1]

-8 (A1)

(b) (i) Solve $x^2 - 2x - 8 = 0$.

[3]

$x^2 - 2x - 8 = 0$

$(x+2)(x-4) = 0$ Cross method (M1)

$x + 2 = 0$ or $x - 4 = 0$

$x = -2$ or $x = 4$ (A1)(A1)

(ii) Hence, express the quadratic function in the form $y = a(x-p)(x-q)$, $p \le q$.

[1]

$y = (x+2)(x-4)$ (A1)

The quadratic function can be expressed in the form $y = a(x-h)^2 + k$.

(c) (i) Find h.

[2]

h

$= \dfrac{-2+4}{2}$ $h = \dfrac{p+q}{2}$ (M1)

$= 1$ (A1)

(ii) Hence, find k.

[2]

k

$= 1^2 - 2(1) - 8$ $k = ah^2 + bh + c$ (M1)

$= -9$ (A1)

(iii) Write down the equation of the axis of symmetry of the quadratic graph.

[1]

$x = 1$ (A1)

Exercise 2.1

A quadratic function is defined as $y = x^2 - 16x$, $x \in \mathbb{R}$.

(a) Write down the y -intercept of the quadratic graph.

[1]

(b) (i) Solve $x^2 - 16x = 0$.

[3]

(ii) Hence, express the quadratic function in the form $y = a(x-p)(x-q)$, $p \le q$.

[1]

The quadratic function can be expressed in the form $y = a(x-h)^2 + k$.

(c) (i) Find h .

[2]

(ii) Hence, find k .

[2]

(iii) Write down the equation of the axis of symmetry of the quadratic graph.

[1]

Example 2.2

Consider the graphs of $y = x^2 + 4kx + 9$ and $y = 2kx - 7$, $k \in \mathbb{R}$.

(a) Find the set of values of $k \in \mathbb{R}$ such that the two graphs have no intersection points.

[5]

$$\begin{cases} y = x^2 + 4kx + 9 \\ y = 2kx - 7 \end{cases}$$

$\therefore x^2 + 4kx + 9 = 2kx - 7$

$x^2 + 2kx + 16 = 0$ — $x^2 + 2kx + 16 = 0$ (A1)

The two graphs have no intersection points.

$\therefore \Delta < 0$ — $\Delta < 0$ (R1)

$(2k)^2 - 4(1)(16) < 0$ — $\Delta = b^2 - 4ac$ (M1)

$4k^2 - 64 < 0$ — $4k^2 - 64 < 0$ (A1)

$4k^2 < 64$

$k^2 < 16$

$\therefore -4 < k < 4$ — (A1)

Consider the case when $k = 8$. The x-coordinates of the points of intersection can be expressed as $x = m \pm \sqrt{48}$, $m \in \mathbb{Z}$.

(b) Find m.

[2]

$x^2 + 2(8)x + 16 = 0$

$x^2 + 16x + 16 = 0$

$x = \dfrac{-16 \pm \sqrt{16^2 - 4(1)(16)}}{2(1)}$ — $x = \dfrac{-b \pm \sqrt{b^2 - 4ac}}{2a}$ (M1)

$x = \dfrac{-16 \pm \sqrt{192}}{2}$

$x = \dfrac{-16 \pm 2\sqrt{48}}{2}$

$x = -8 \pm \sqrt{48}$

$\therefore m = -8$ — (A1)

Exercise 2.2

Consider the graphs of $y = x^2 + 2kx + 5$ and $y = 3kx - 4$, $k \in \mathbb{R}$.

(a) Find the set of values of $k \in \mathbb{R}$ such that the two graphs have two intersection points.

[5]

Consider the case when $k = 7$. The x-coordinates of the points of intersection can be expressed as $x = \dfrac{m \pm \sqrt{r}}{2}$, m, $r \in \mathbb{Z}$.

(b) Find $m + r$.

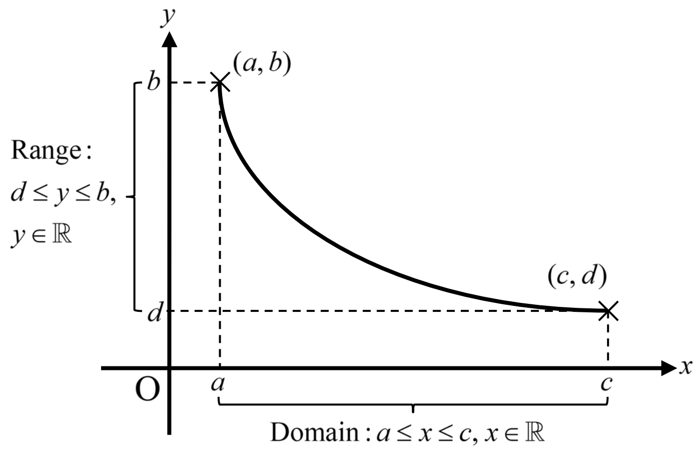
[2]

2 Functions

Important Notes

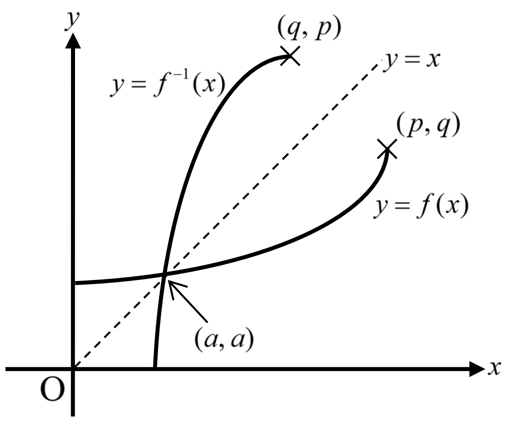
Notations related to a general function $y = f(x)$:

1. $f(a)$: Functional value (value of y) when $x = a$
2. Domain: Set of all possible values of x
3. Range: Set of all possible values of y
4. Root(s) of the equation $f(x) = 0$: x-intercept(s) of the graph of the corresponding function $y = f(x)$, which is equivalent to the zero(s) of $y = f(x)$
5. $(f \circ g)(x) = f(g(x))$: Composite function when $g(x)$ is substituted into $f(x)$



Properties of $y = f^{-1}(x)$:

1. Domain of f^{-1} is consistent with range of f
2. Range of f^{-1} is consistent with domain of f
3. $f(f^{-1}(x)) = f^{-1}(f(x)) = x$
4. Graph of $y = f^{-1}(x)$: Reflection of the graph of $y = f(x)$ about $y = x$
5. The points of intersection of the graphs of f^{-1} and f lies on $y = x$
6. $y = f^{-1}(x)$ exists only when $y = f(x)$ is one-to-one in the restricted domain

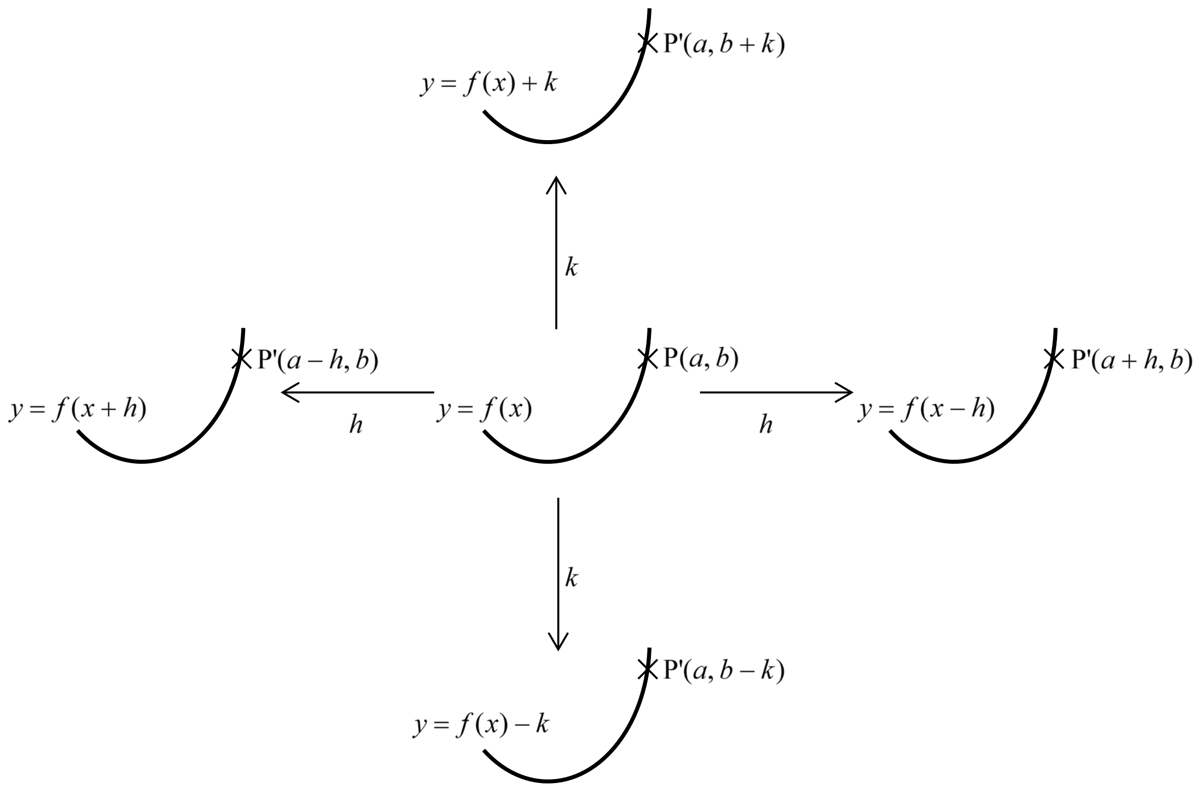


Steps of finding an expression of the inverse function $y = f^{-1}(x)$ from $y = f(x)$:

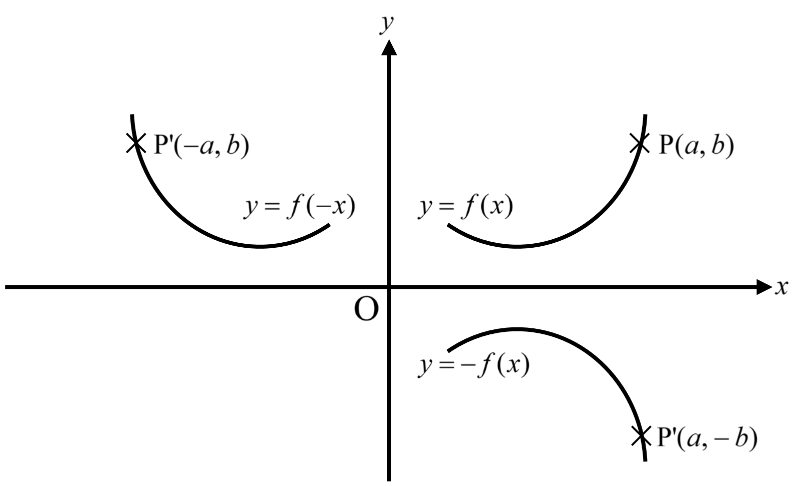
1. Start from expressing y in terms of x
2. Interchange x and y
3. Make y the subject in terms of x

Summary of the transformations of functions:

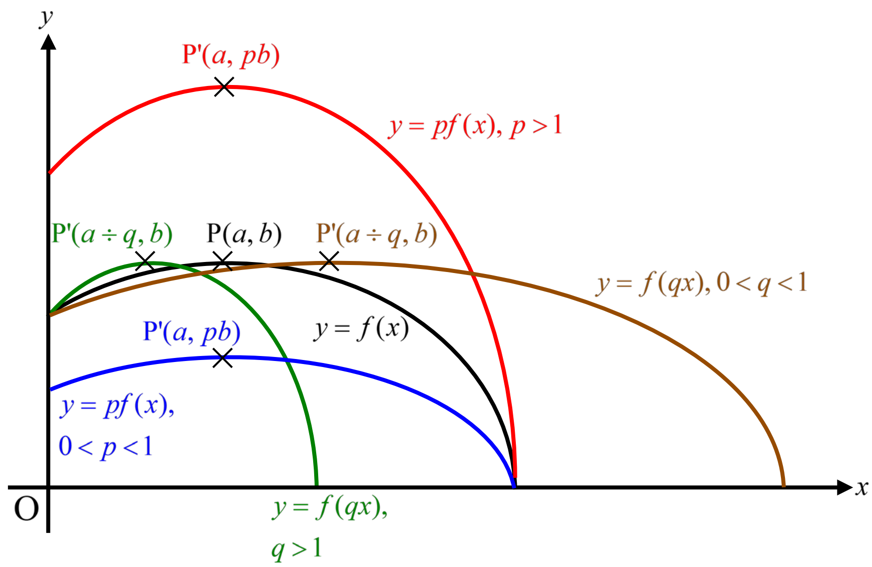
1. $y = f(x) \rightarrow y = f(x) + k$: Translate upward by k units
2. $y = f(x) \rightarrow y = f(x) - k$: Translate downward by k units
3. $y = f(x) \rightarrow y = f(x - h)$: Translate to the right by h units
4. $y = f(x) \rightarrow y = f(x + h)$: Translate to the left by h units



5. $y = f(x) \rightarrow y = -f(x)$: Reflection about the x-axis
6. $y = f(x) \rightarrow y = f(-x)$: Reflection about the y-axis



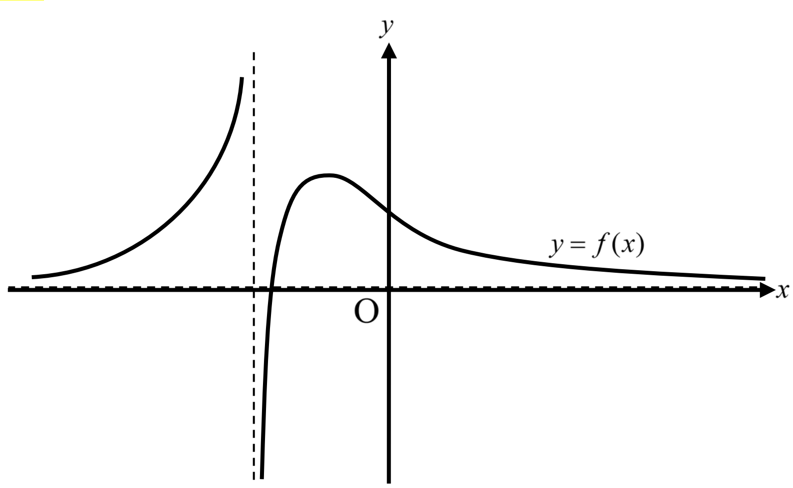
7. $y = f(x) \to y = p\,f(x)$: Vertical stretch of scale factor p, $p > 1$ (compression for $0 < p < 1$)
8. $y = f(x) \to y = f(q\,x)$: Horizontal compression of scale factor q, $q > 1$ (stretch for $0 < q < 1$)



9. $\begin{pmatrix} h \\ k \end{pmatrix}$: Composite translation vector of h units to the right and k units upward

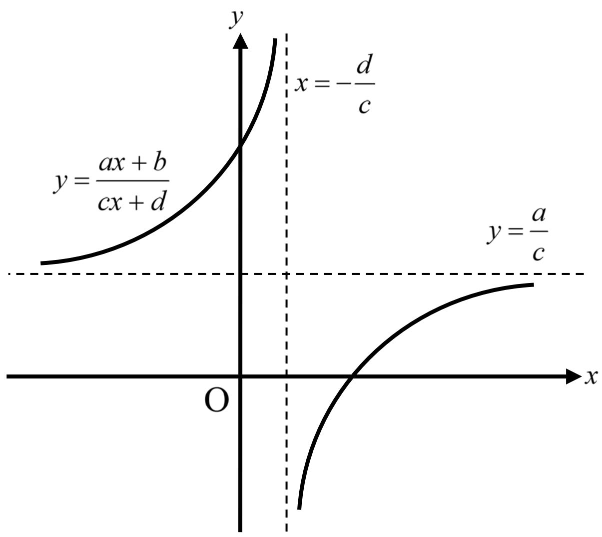
Types of asymptotes of the graph of $y = f(x)$:

1. Vertical asymptote: The vertical boundary where $f(x)$ is undefined
2. The equation of the vertical asymptote can be found by considering the denominator expression of $f(x)$ equals to zero
3. Horizontal asymptote: The horizontal boundary (level) where y approaches when x tends to positive/negative infinity
4. The equation of the horizontal asymptote can be found by considering $y = \lim_{x \to \infty} f(x)$



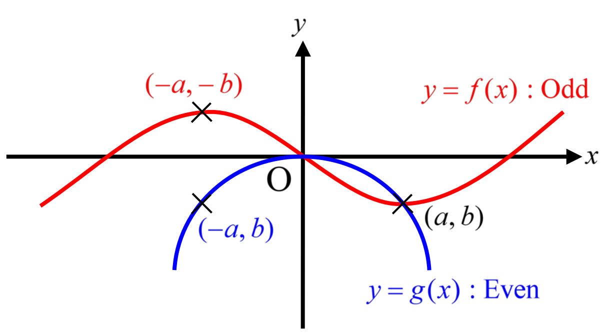
Properties of the rational function $y=\dfrac{ax+b}{cx+d}$, $a,b,c,d\in\mathbb{R}$, $c\neq 0$:

1. $y=\dfrac{1}{x}$: Reciprocal function
2. $y=\dfrac{a}{c}$: Horizontal asymptote
3. $x=-\dfrac{d}{c}$: Vertical asymptote from $cx+d=0$
4. Substitute $y=0$ and make x the subject to find the x-intercept
5. Substitute $x=0$ and make y the subject to find the y-intercept



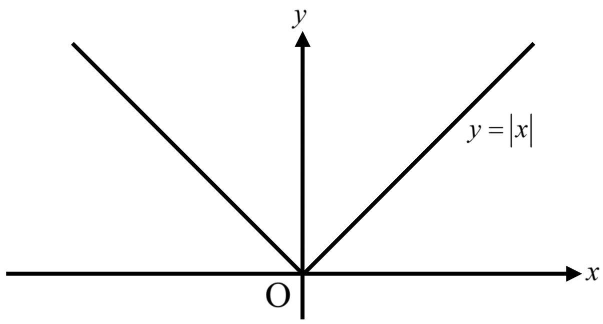
Odd and even functions:

1. $y=f(x)$ is odd if $f(-x)=-f(x)$
2. The graph of an odd function is symmetric about the origin
3. $y=f(x)$ is even if $f(-x)=f(x)$
4. The graph of an even function is symmetric about the y-axis

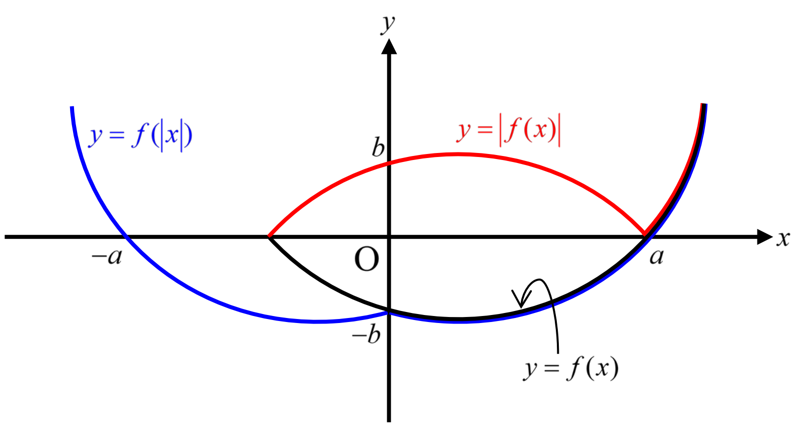


Absolute sign and the corresponding transformations:

1. $|x| = \begin{cases} x \text{ if } x \geq 0 \\ -x \text{ if } x < 0 \end{cases}$: Absolute function



2. $y = f(x) \to y = |f(x)|$: Reflection about the x-axis only for the part of the original graph of f which was below the x-axis, combined with the copy of the original graph of f which was above the x-axis
3. $y = f(x) \to y = f(|x|)$: Reflection about the y-axis only for the part of the original graph of f which was on the right-hand side of the y-axis, combined with the copy of the original graph of f which was on the right-hand side of the y-axis



Notes on GDC		
TEXAS TI-84 Plus CE y= to input the function →2nd window to set the starting row to be at least 1000 →2nd graph to look at the function values when x is at least 1000 to find the equation of the horizontal asymptote of the graph	**TEXAS TI-Nspire CX** Graph to input the function to generate a table →ctrl T to generate a table →menu 2 5 to set the starting row to look at the function values when x is at least 1000 to find the equation of the horizontal asymptote of the graph	**CASIO fx-CG50** Table to input the function to generate a table →F5 to set the starting row to be at least 1000 →F6 to look at the function values when x is at least 1000 to find the equation of the horizontal asymptote of the graph

Example 2.3

The function f is defined as $f(x)=\dfrac{2x+4}{7x-7}$, $x\neq 1$, $x\in\mathbb{R}$.

(a) Find

(i) the zero of $f(x)$;

[2]

$f(x)=0$ — $f(x)=0$ (M1)

$\dfrac{2x+4}{7x-7}=0$

$2x+4=0$

$2x=-4$

$x=-2$ — (A1)

(ii) the y-intercept of the graph of $f(x)=\dfrac{2x+4}{7x-7}$.

[2]

The y-intercept

$=\dfrac{2(0)+4}{7(0)-7}$ — Substitute $x=0$ (M1)

$=-\dfrac{4}{7}$ — (A1)

(b) For the graph of f, write down

(i) the equation of the vertical asymptote;

[1]

$x=1$ — (A1)

(ii) the equation of the horizontal asymptote;

[1]

$y=\dfrac{2}{7}$ — (A1)

(iii) the range of f;

[1]

$y\neq\dfrac{2}{7}$, $y\in\mathbb{R}$ — (A1)

Let $f^{-1}(x)$ be the inverse function of $f(x)$.

(c) (i) Write down the range of $f^{-1}(x)$.

[1]

$y \neq 1,\ y \in \mathbb{R}$ (A1)

(ii) Find an expression of $f^{-1}(x)$.

[3]

$y = \dfrac{2x+4}{7x-7}$

$\rightarrow x = \dfrac{2y+4}{7y-7}$ Interchange x and y (M1)

$x(7y-7) = 2y+4$

$7xy - 7x = 2y+4$

$7xy - 2y = 7x+4$ Combine like terms (M1)

$y(7x-2) = 7x+4$

$y = \dfrac{7x+4}{7x-2}$

$\therefore f^{-1}(x) = \dfrac{7x+4}{7x-2}$ (A1)

It is given that $g(x) = 3x+1$, $x \neq 0$, $x \in \mathbb{R}$ and $(f \circ g)(x) = \dfrac{2}{7} + \dfrac{a}{x}$, $a \in \mathbb{Q}$.

(d) Find a.

[2]

$(f \circ g)(x)$

$= f(g(x))$

$= \dfrac{2(3x+1)+4}{7(3x+1)-7}$ $\dfrac{2(3x+1)+4}{7(3x+1)-7}$ (M1)

$= \dfrac{6x+6}{21x}$

$= \dfrac{2}{7} + \dfrac{2}{7x}$

$\therefore a = \dfrac{2}{7}$ (A1)

Exercise 2.3

The function f is defined as $f(x) = \dfrac{x-3}{2x+4}$, $x \neq -2$, $x \in \mathbb{R}$.

(a) Find

(i) the zero of $f(x)$;

[2]

(ii) the y-intercept of the graph of $f(x) = \dfrac{x-3}{2x+4}$.

[2]

(b) For the graph of f, write down

(i) the equation of the vertical asymptote;

[1]

(ii) the equation of the horizontal asymptote;

[1]

(iii) the range of f;

[1]

Let $f^{-1}(x)$ be the inverse function of $f(x)$.

(c) (i) Write down the range of $f^{-1}(x)$.

[1]

(ii) Find an expression of $f^{-1}(x)$.

[3]

It is given that $g(x) = 0.5 - 5x$, $x \neq 0$, $x \in \mathbb{R}$ and $(f^{-1} \circ g)(x) = \frac{a}{x} - 2$, $a \in \mathbb{Q}$.

(d) Find a.

[2]

Example 2.4

Let $f(x)=(x+3)^2$ and $g(x)=2x^2$. The graph of g can be obtained from the graph of f using two transformations.

(a) Give a full geometric description of each of the two transformations.

[2]

Translate to the right by 3 units (A1)
followed by a vertical stretch of scale factor 2 (A1)

The graph of g is then reflected about the y-axis, followed by a translation by the vector $\begin{pmatrix} 3 \\ -2 \end{pmatrix}$ to give the graph of h.

(b) Find an expression of $h(x)$.

[3]

$h(x)$
$=g(-(x-3))-2$ $\quad h(x)=g(-(x-3))-2$ (M1)
$=g(-x+3)-2$
$=2(-x+3)^2-2$ $\quad g(-x+3)=2(-x+3)^2$ (A1)
$=2(x^2-6x+9)-2$
$=2x^2-12x+16$ (A1)

The point $(-1,4)$ on the graph of f is translated to the point P on the graph of h.

(c) Find the coordinates of P.

[5]

The image after transformed to g
$=(-1+3,4\times 2)$ $\quad x+3$ (A1) & $2y$ (A1)
$=(2,8)$
The coordinates of P
$=(-2+3,8-2)$ $\quad -x+3$ (A1) & $y-2$ (A1)
$=(1,6)$ (A1)

Exercise 2.4

Let $f(x) = -x^2$ and $g(x) = x^2 + 5$. The graph of g can be obtained from the graph of f using two transformations.

(a) Give a full geometric description of each of the two transformations.

[2]

The graph of g is then compressed horizontally of the scale factor 3, followed by a translation by the vector $\begin{pmatrix} -4 \\ 0 \end{pmatrix}$ to give the graph of h.

(b) Find an expression of $h(x)$.

[3]

The point $(-3, 9)$ on the graph of f is translated to the point P on the graph of h.

(c) Find the coordinates of P.

[5]

Example 2.5

Consider the function $f(x)=2x\sqrt{1-4x^2}$, $-0.5\le x\le 0.5$.

(a) Show that f is an odd function.

[2]

$f(-x)$

$=2(-x)\sqrt{1-4(-x)^2}$ — $f(-x)$ (A1)

$=-2x\sqrt{1-4x^2}$

$=-f(x)$ — (A1)

Thus, f is an odd function. — (AG)

(b) Find the range of f .

[2]

By considering the graph of $y=2x\sqrt{1-4x^2}$, the coordinates of the maximum point and the minimum point are $(0.3535545, 0.5)$ and $(-0.3535545, -0.5)$ respectively. — GDC approach (M1)

Thus, the range of f is $-0.5\le y\le 0.5$, $y\in\mathbb{R}$. — (A1)

(c) Write down the restricted domain of f such that f^{-1} exists.

[2]

$-0.354\le x\le 0.354$, $x\in\mathbb{R}$ — -0.354 (A1) & 0.354 (A1)

(d) Solve the inequality $|f(x)|>x$.

[3]

$|f(x)|>x$

$\therefore \left|2x\sqrt{1-4x^2}\right|>x$

$\left|2x\sqrt{1-4x^2}\right|-x>0$ — Correct inequality (A1)

By considering the graph of $y=\left|2x\sqrt{1-4x^2}\right|-x$,

the graph is above the horizontal axis when $-0.5\le x<0$ or $0<x<0.4330127$. — GDC approach (M1)

$\therefore -0.5\le x<0$ or $0<x<0.433$ — (A1)

Exercise 2.5

Consider the function $f(x) = -x^2\sqrt{9-x^2}$, $-3 < x < 3$.

(a) Show that f is an even function.

[2]

(b) Find the range of f .

[2]

It is given that the restricted domain of f , such that f^{-1} exists, is defined as $c \le x \le 0$, x , $c \in \mathbb{R}$

(c) Write down c .

[1]

(d) Solve the inequality $|x| - 2 \ge f(x)$.

[3]

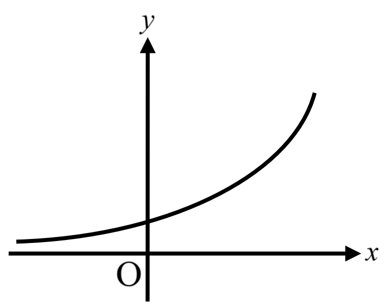
3 Exponential and Logarithmic Functions

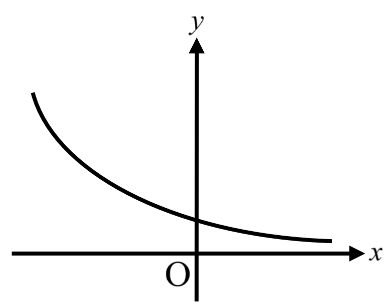
Important Notes

Exponential function: A function with x to be the power (exponent) of a positive real number other than 1

Properties of an exponential function in the form $y = a^x$, base $a \in \mathbb{R}^+$

1. $a > 1$: Exponentially increase | $0 < a < 1$: Exponentially decrease

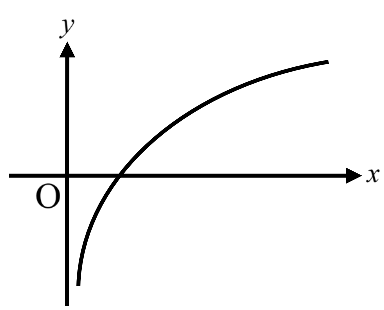


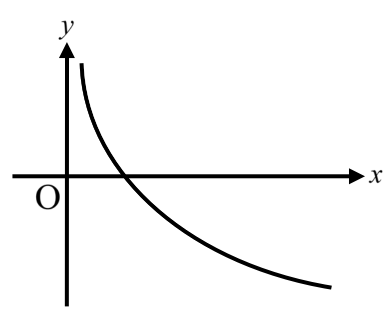


2. $a^0 = 1$: y-intercept of the graph
3. $x \in \mathbb{R}$: Domain of $y = a^x$
4. $y > 0$, $y \in \mathbb{R}$: Range of $y = a^x$
5. $y = 0$: Equation of the horizontal asymptote of the graph

Properties of a logarithmic function in the form $y = \log_a x$, base $a \in \mathbb{R}^+$

1. $y = \log_a x$ is the inverse function of $y = a^x$
2. $a > 1$: Increase | $0 < a < 1$: Decrease





3. 1: x-intercept of the graph
4. $x > 0$, $x \in \mathbb{R}$: Domain of $y = \log_a x$
5. $y \in \mathbb{R}$: Range of $y = \log_a x$
6. $x = 0$: Equation of the vertical asymptote of the graph

7. $y = \log x (= \log_{10} x)$: Logarithmic function of the common base (base 10)
8. $y = \ln x (= \log_e x)$: Natural logarithmic function of the base e, where $e = 2.718281828\cdots$ is the exponential number

Laws of logarithm, where $a, b, c, p, q, x > 0$:

1. $b = a^x \Leftrightarrow x = \log_a b$
2. $1 = a^0 \Leftrightarrow 0 = \log_a 1$
3. $a = a^1 \Leftrightarrow 1 = \log_a a$
4. $\log_a p + \log_a q = \log_a (pq)$
5. $\log_a p - \log_a q = \log_a \left(\frac{p}{q}\right)$
6. $\log_a p^n = n \log_a p$
7. $\log_b a = \frac{\log_c a}{\log_c b}$: Change of base formula

Methods of solving an exponential equation $a^x = b$, $a \in \mathbb{R}^+$:

1. Change b into a^y such that $a^x = a^y \Rightarrow x = y$
2. Take logarithm for both sides

Example 2.6

(a) Express and simplify the following in terms of $\log_2 x$, $x \in \mathbb{R}^+$:

(i) $\log_2 x^5$;

[1]

$\log_2 x^5 = 5\log_2 x$ (A1)

(ii) $\log_2 (8x)$;

[3]

$\log_2 (8x)$

$= \log_2 8 + \log_2 x$ — $\log_2 (pq) = \log_2 p + \log_2 q$ (M1)

$= \log_2 2^3 + \log_2 x$ — $8 = 2^3$ (M1)

$= 3\log_2 2 + \log_2 x$

$= 3 + \log_2 x$ (A1)

(iii) $\ln\left(\frac{x}{e}\right)$.

[3]

$\ln\left(\frac{x}{e}\right)$

$= \ln x - \ln e$ — $\ln\left(\frac{p}{q}\right) = \ln p - \ln q$ (M1)

$= \ln x - 1$ — $\ln e = 1$ (M1)

$= \frac{\log_2 x}{\log_2 e} - 1$ (A1)

(b) Hence, solve the equation

$$\log_2 x^5 - \log_2(8x) + \left(1 + \ln\left(\frac{x}{e}\right)\right)\log_2 e = 3\log_2 x + 5, \ x > 0.$$

[4]

$$\log_2 x^5 - \log_2(8x) + \left(1 + \ln\left(\frac{x}{e}\right)\right)\log_2 e = 3\log_2 x + 5$$

$$\therefore 5\log_2 x - (3 + \log_2 x) + \left(\frac{\log_2 x}{\log_2 e}\right)\log_2 e$$ Substitution (M1)

$$= 3\log_2 x + 5$$

$$5\log_2 x - 3 - \log_2 x + \log_2 x = 3\log_2 x + 5$$

$$2\log_2 x = 8$$ Combine like terms (M1)

$$\log_2 x = 4$$

$$\therefore x = 2^4$$ $x = \log_a b \Leftrightarrow b = a^x$ (M1)

$$x = 16$$ (A1)

Exercise 2.6

(a) Express and simplify the following in terms of $\log_3 x$ and/or $\log_3 2$, $x \in \mathbb{R}^+$:

(i) $\log_3 x^6$; [1]

(ii) $\log_3(16x)$; [3]

(iii) $\log_2 3$. [2]

(b) Hence, solve the equation $\frac{2}{3}\log_3 x^6 + \frac{1}{\log_2 3} + \log_3(16x) = 0$, $x > 0$. [4]

Example 2.7

A town is concerned about pollution, and decides to look at the number of people using private cars. At the end of 2023, there were 500 private cars in the town. After t years the number of private cars, N, in the city is given by $N = N_0 e^{kt}$, N_0, $k > 0$, $t \geq 0$.

(a) Show that $N_0 = 500$.

[1]

$500 = N_0 e^{k(0)}$ — $N = 500$ & $t = 0$ (A1)

$N_0 = 500$ — (AG)

There are 710 private cars at the end of 2026.

(b) Find k.

[3]

$710 = 500e^{k(3)}$ — $N = 710$ & $t = 3$ (A1)

$500e^{3k} - 710 = 0$

By considering the graph of $y = 500e^{3k} - 710$,

the horizontal intercept is 0.1168856. — GDC approach (M1)

$\therefore k = 0.117$ — (A1)

(c) Find the year in which the number of private cars is triple the number of private cars there were at the end of 2023.

[3]

$500(3) = 500e^{0.1168856t}$ — Correct equation (A1)

$3 = e^{0.1168856t}$

$e^{0.1168856t} - 3 = 0$

By considering the graph of $y = e^{0.1168856t} - 3$,

the horizontal intercept is 9.3990388. — GDC approach (M1)

$\therefore$ The required year is 2033. — (A1)

Exercise 2.7

A population of Bulbul birds, P_t, can be modelled by the equation $P_t = P_0 e^{kt}$, where P_0 is the initial population, and t is measured in decades. After one decade (ten years), it is estimated that the population is 10% less than the initial population.

(a) Find k, correct the answer to four decimal places.

[3]

(b) Hence, interpret the meaning of the value of k.

[1]

(c) Find the least number of complete years such that the population is half of the initial population.

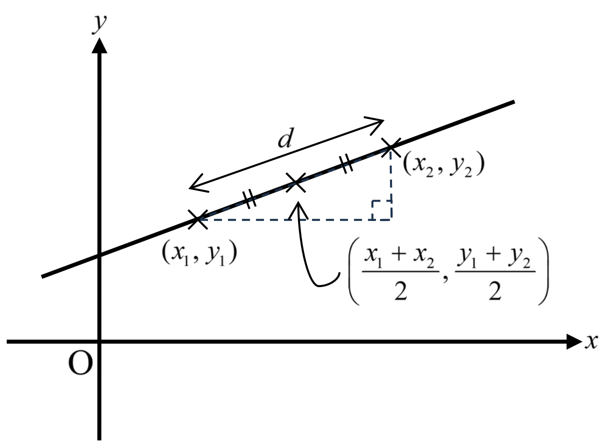
[3]

4 Equations of Straight Lines

Important Notes

Consider any two points $\mathrm{P}(x_1, y_1)$ and $\mathrm{Q}(x_2, y_2)$ on a $x-y$ plane:

1. $m=\dfrac{y_2-y_1}{x_2-x_1}$: Slope (gradient) of PQ
2. $d=\sqrt{(x_2-x_1)^2+(y_2-y_1)^2}$: Distance between P and Q
3. $\left(\dfrac{x_1+x_2}{2}, \dfrac{y_1+y_2}{2}\right)$: The mid-point of PQ



Consider any two straight lines L_1 and L_2 with corresponding slopes m_1 and m_2 respectively:

1. $m_1=m_2$ if L_1 and L_2 are parallel ($L_1 // L_2$)
2. $m_1 \times m_2=-1$ if L_1 and L_2 are perpendicular ($L_1 \perp L_2$)

$y-y_1=m(x-x_1)$: The point-slope formula to find the equation of a straight line with slope m and a fixed point (x_1, y_1) on the line

Forms of equations of straight lines:

1. $y=mx+c$: Slope-intercept form with slope m and y-intercept c
2. $Ax+By+C=0$: General form, where $A \in \mathbb{Z}^+$, $B, C \in \mathbb{Z}$

Axes intercepts of a straight line:

1. Substitute $y=0$ and make x the subject to find the x-intercept
2. Substitute $x=0$ and make y the subject to find the y-intercept

Example 2.8

A line joins the points $\mathrm{A}(10, 3)$ and $\mathrm{B}(-2, -7)$.

(a) Find the gradient of the line AB.

[2]

The gradient

$= \frac{-7-3}{-2-10}$ $\quad m = \frac{y_2 - y_1}{x_2 - x_1}$ (M1)

$= \frac{5}{6}$ (A1)

Let M be the midpoint of the line AB.

(b) (i) Write down the coordinates of M.

[1]

$(4, -2)$ (A1)

(ii) Hence, find the exact distance between A and M.

[2]

The exact distance

$= \sqrt{(4-10)^2 + (-2-3)^2}$ $\quad d = \sqrt{(x_2 - x_1)^2 + (y_2 - y_1)^2}$ (M1)

$= \sqrt{61}$ (A1)

(c) Find the equation of the line perpendicular to AB and passing through M, giving the answer in slope-intercept form.

[3]

The required slope

$= -1 \div \frac{5}{6}$ $\quad m_1 \times m_2 = -1$ (M1)

$= -\frac{6}{5}$

The equation:

$y - (-2) = -\frac{6}{5}(x-4)$ $\quad y - y_1 = m(x - x_1)$ (M1)

$y + 2 = -\frac{6}{5}x + \frac{24}{5}$

$y = -\frac{6}{5}x + \frac{14}{5}$ (A1)

Exercise 2.8

A line joins the points $\mathrm{A}(0,-9)$ and $\mathrm{B}(-8,1)$.

(a) Find the gradient of the line AB.

[2]

Let M be the midpoint of the line AB.

(b) (i) Write down the coordinates of M.

[1]

(ii) Hence, find the exact distance between B and M.

[2]

(c) Find the equation of the line perpendicular to AB and passing through B, giving the answer in general form.

[3]

5 Polynomials

Important Notes

$f(x)=a_nx^n+a_{n-1}x^{n-1}+\cdots+a_2x^2+a_1x+a_0$: The nth degree polynomial in which the coefficient of x^r is a_r, $a_r\in\mathbb{R}$, $r=0,1,2,\cdots,n$, with $a_n\neq 0$

Properties of a polynomial equation $f(x)=a_nx^n+a_{n-1}x^{n-1}+\cdots+a_2x^2+a_1x+a_0=0$:

1. The total number of real and complex roots of $f(x)=0$ is n
2. The maximum number of real roots of $f(x)=0$ is n
3. $r_1, r_2, \cdots, r_n$: Roots
4. $r_1+r_2+\cdots+r_n=-\frac{a_{n-1}}{a_n}$: Sum of n roots
5. $r_1r_2r_3\cdots r_{n-1}r_n=(-1)^n\frac{a_0}{a_n}$: Product of n roots

Remainder and factor theorem:

1. $f(a)$ is the remainder when $f(x)$ is divided by $(x-a)$
2. $f\left(\frac{q}{p}\right)$ is the remainder when $f(x)$ is divided by $(px-q)$
3. $(x-a)$ is a factor of $f(x)$ if $f(a)=0$
4. $(px-q)$ is a factor of $f(x)$ if $f\left(\frac{q}{p}\right)=0$

Partial fractions:

1. $\frac{ax+b}{(cx+d)(ex+f)}$ can be expressed as $\frac{P}{cx+d}+\frac{Q}{ex+f}$, $a,b,c,d,e,f,P,Q\in\mathbb{R}$
2. $\frac{ax+b}{(cx+d)^2}$ can be expressed as $\frac{P}{cx+d}+\frac{Q}{(cx+d)^2}$, $a,b,c,d,P,Q\in\mathbb{R}$

Properties of the function $f(x)=\dfrac{ax+b}{(cx+d)(ex+f)}$, $a, b, c, d, e, f \in \mathbb{R}$

1. Vertical asymptote: $x=-\dfrac{d}{c}$, $x=-\dfrac{f}{e}$
2. Horizontal asymptote: $y=0$

Properties of the function $f(x)=\dfrac{ax^2+bx+c}{dx+e}$, $a, b, c, d, e \in \mathbb{R}$

1. Vertical asymptote: $x=-\dfrac{e}{d}$
2. $\dfrac{ax^2+bx+c}{dx+e}$ can be expressed as $Ax+B+\dfrac{C}{dx+e}$, $A, B, C \in \mathbb{R}$
3. Oblique asymptote: $y=Ax+B$, an asymptote which is neither vertical nor horizontal

Example 2.9

The same remainder is found when $f(x)=x^3-3x^2+kx-3k$ and $g(x)=x^3-9x^2+mx-24$, $k, m\in\mathbb{R}$ are divided by $x-3$.

(a) Find m.

[4]

$f(3)$	
$=3^3-3(3)^2+k(3)-3k$	
$=0$	0 (A1)
$g(3)$	
$=3^3-9(3)^2+m(3)-24$	
$=3m-78$	$3m-78$ (A1)
$f(3)=g(3)$	$f(3)=g(3)$ (M1)
$\therefore 0=3m-78$	
$78=3m$	
$m=26$	(A1)

It is given that one of the real roots of $f(x)=0$ is 4.

(b) Find k.

[5]

$\because f(3)=0$	
$\therefore 3$ and 4 are two real roots of $f(x)=0$	Two known roots (M1)
Let α be the third root.	
Sum of roots $=-\frac{-3}{1}$	$r_1+r_2+r_3=-\frac{a_2}{a_3}$ (A1)
$\alpha+3+4=3$	
$\alpha=-4$	$\alpha=-4$ (A1)
Product of roots $=(-1)^3\frac{-3k}{1}$	$r_1r_2r_3=(-1)^3\frac{a_0}{a_3}$ (A1)
$(-4)(3)(4)=3k$	
$k=-16$	(A1)

Exercise 2.9

The same remainder is found when $f(x)=2x^3+kx^2+6x+8$ and $g(x)=x^4-6x^2-5kx-3k$, $k\in\mathbb{R}$ are divided by $x+1$.

(a) Find k.

[4]

It is given that the only one real root of $f(x)=0$ is -2.

(b) Find the sum and the product of the two complex roots.

[4]

Example 2.10

The function f is defined as $f(x)=\frac{x+3}{2x^2-3x+1}$, where $x\neq\frac{1}{2}$, $x\neq1$, $x\in\mathbb{R}$.

(a) Write down

(i) the equations of the vertical asymptotes of the graph of f.

[2]

$x=\frac{1}{2}$, $x=1$ (A1)(A1)

(ii) the axes intercepts of the graph of f.

[2]

x-intercept $=-3$ (A1)

y-intercept $=3$ (A1)

(b) Find the coordinates of the local maximum point of the graph of f.

[2]

By considering the graph of $y=\frac{x+3}{2x^2-3x+1}$,

the coordinates of the maximum point

are $(0.7416593, -29.96663)$. GDC approach (M1)

Thus, the coordinates are $(0.742, -30.0)$. (A1)

(c) Find the range of f.

[2]

By considering the graph of $y=\frac{x+3}{2x^2-3x+1}$,

the coordinates of the minimum point

are $(-4.999995, -0.030303)$. GDC approach (M1)

Thus, the range of f is

$y\leq-30.0$ or $y\geq-0.0303$, $y\in\mathbb{R}$. $y\leq-30.0$ (A1) & $y\geq-0.0303$ (A1)

$f(x)$ can be expressed as $\frac{A}{2x-1}+\frac{B}{x-1}$, $A, B \in \mathbb{R}$.

(d) Find the values of A and B.

[5]

Let $\frac{x+3}{2x^2-3x+1}=\frac{A}{2x-1}+\frac{B}{x-1}$.

$\frac{x+3}{2x^2-3x+1}=\frac{A(x-1)}{(2x-1)(x-1)}+\frac{B(2x-1)}{(x-1)(2x-1)}$

$x+3=Ax-A+2Bx-B$ Expansion (M1)

$\therefore \begin{cases} x=Ax+2Bx \\ 3=-A-B \end{cases}$ Compare coefficients (M1)

$1=A+2B$

$A=1-2B$

$3=-A-B$

$\therefore 3=-(1-2B)-B$ Substitute $A=1-2B$ (M1)

$3=-1+2B-B$

$B=4$ (A1)

$\therefore A=1-2(4)$

$A=-7$ (A1)

The function g is defined as $g(x)=\frac{1}{f(x)}$, where $x \neq -3$, $x \in \mathbb{R}$.

(e) Write down

(i) the equation of the vertical asymptote of the graph of g.

[1]

$x=-3$ (A1)

(ii) the axes intercepts of the graph of g.

[2]

x-intercepts $=\frac{1}{2}$ or 1 (A1)

y-intercept $=\frac{1}{3}$ (A1)

(f) Find the equation of the oblique asymptote of the graph of g.

[4]

$$g(x)=\frac{2x^2-3x+1}{x+3}$$

Let $\frac{2x^2-3x+1}{x+3}=2x+C+\frac{D}{x+3}$. $2x+A$ (A1)

$$\frac{2x^2-3x+1}{x+3}=\frac{(2x+C)(x+3)}{x+3}+\frac{D}{x+3}$$

$2x^2-3x+1=2x^2+6x+Cx+3C+D$ Expansion (M1)

$\therefore\begin{cases}-3x=6x+Cx\\1=3C+D\end{cases}$ Compare coefficients (M1)

$-3=6+C$

$C=-9$

Thus, the equation is $y=2x-9$. (A1)

(g) Find the range of g.

[3]

By considering the graph of $y=\frac{2x^2-3x+1}{x+3}$,

the coordinates of the maximum and the minimum point are $(-6.741657,-29.96663)$ and $(0.7416573,-0.03337)$ respectively. GDC approach (M1)

Thus, the range of g is

$y\le-30.0$ or $y\ge-0.0334$, $y\in\mathbb{R}$. $y\le-30.0$ (A1) & $y\ge-0.0334$ (A1)

Exercise 2.10

The function f is defined as $f(x)=\dfrac{x+5}{x^2-6x+8}$, where $x \neq 2$, $x \neq 4$, $x \in \mathbb{R}$.

(a) Write down

(i) the equations of the vertical asymptotes of the graph of f.

[2]

(ii) the axes intercepts of the graph of f.

[2]

(b) Find the coordinates of the local minimum point of the graph of f.

[2]

(c) Find the range of f.

[2]

$f(x)$ can be expressed as $\dfrac{A}{x-2}+\dfrac{B}{x-4}$, $A, B \in \mathbb{R}$.

(d) Find the values of A and B.

[5]

The function g is defined as $g(x)=\frac{1}{f(x)}$, where $x\neq -5$, $x\in\mathbb{R}$.

(e) Write down

(i) the equation of the vertical asymptote of the graph of g.

[1]

(ii) the axes intercepts of the graph of g.

[2]

(f) Find the equation of the oblique asymptote of the graph of g.

[4]

(g) Find the range of g.

[3]