# AA HL Practice Set 2 Paper 3 Solution

**1.** (a) arc $P_1B$

$= \frac{1}{4}\pi(1)^2$ (M1) for valid approach

$= \frac{1}{4}\pi$ A1

[2]

(b) (i) $1$ A1

(ii) $\sqrt{2}$ A1

[2]

(c) (i) $R(3)$

$= 3\left(\frac{1}{2}(\text{OA})(\text{OP}_1)\sin \text{A}\hat{\text{O}}\text{P}_1\right)$ (M1) for valid approach

$= 3\left(\frac{1}{2}(1)(1)\sin \frac{180^\circ}{3}\right)$ (A1) for substitution

$= \frac{3}{2}\sin 60^\circ$ A1

(ii) $\text{AP}_1 = \text{OP}_1$ as $\text{AOP}_1$ is an equilateral triangle. R1

[4]

(d) $R(4)$

$= 4\left(\frac{1}{2}(\text{OA})(\text{OP}_1)\sin \text{A}\hat{\text{O}}\text{P}_1\right)$ M1

$= 4\left(\frac{1}{2}(1)(1)\sin \frac{180^\circ}{4}\right)$ A1

$= 2\sin 45^\circ$ AG

[2]

(e) $AP_1^2 = OA^2 + OP_1^2 - 2(OA)(OP_1)\cos A\hat{O}P_1$ M1

$L(4)^2 = 1^2 + 1^2 - 2(1)(1)\cos 45^\circ$ A1

$L(4)^2 = 2 - 2\left(\frac{\sqrt{2}}{2}\right)$

$L(4)^2 = 2 - \sqrt{2}$ A1

$\because L(4)^4 - 4L(4)^2 + 2$

$= (2-\sqrt{2})^2 - 4(2-\sqrt{2}) + 2$ M1

$= 4 - 4\sqrt{2} + 2 - 8 + 4\sqrt{2} + 2$ M1

$= 0$

Thus, the exact value of $L(4)$ satisfies the equation $x^4 - 4x^2 + 2 = 0$. AG

[5]

(f) (i) $R(n)$

$= n\left(\frac{1}{2}(OA)(OP_1)\sin A\hat{O}P_1\right)$ M1

$= n\left(\frac{1}{2}(1)(1)\sin\frac{180^\circ}{n}\right)$ A1

$= \frac{n}{2}\sin\frac{180^\circ}{n}$ A1

(ii) $\frac{1}{2}\pi$ A1

[4]

(g) (i) $AP_1^2 = OA^2 + OP_1^2 - 2(OA)(OP_1)\cos A\hat{O}P_1$ M1

$L(n)^2 = 1^2 + 1^2 - 2(1)(1)\cos\frac{180^\circ}{n}$ A1

$L(n)^2 = 2 - 2\cos\frac{180^\circ}{n}$

$L(n) = \sqrt{2 - 2\cos\frac{180^\circ}{n}}$ AG

(ii) $\dfrac{L(n)}{R(n)}$

$$= \frac{\sqrt{2-2\cos\frac{180^\circ}{n}}}{\frac{n}{2}\sin\frac{180^\circ}{n}}$$

$$= \frac{\sqrt{2-2\left(1-2\sin^2\frac{90^\circ}{n}\right)}}{\frac{n}{2}\left(2\sin\frac{90^\circ}{n}\cos\frac{90^\circ}{n}\right)}$$ A2

$$= \frac{\sqrt{4\sin^2\frac{90^\circ}{n}}}{n\sin\frac{90^\circ}{n}\cos\frac{90^\circ}{n}}$$ M1

$$= \frac{2\sin\frac{90^\circ}{n}}{n\sin\frac{90^\circ}{n}\cos\frac{90^\circ}{n}}$$ M1

$$= \frac{2}{n\cos\frac{90^\circ}{n}}$$

$$= \frac{2}{n}\sec\frac{90^\circ}{n}$$ AG

[6]

(h) $\dfrac{L(n)}{R(n)} < \dfrac{1}{\pi^\pi}$

$\therefore \dfrac{2}{n}\sec\dfrac{90^\circ}{n} < \dfrac{1}{\pi^\pi}$

$\dfrac{2}{n}\sec\dfrac{90^\circ}{n} - \dfrac{1}{\pi^\pi} < 0$ (A1) for correct inequality

By considering the graph of $y = \dfrac{2}{n}\sec\dfrac{90^\circ}{n} - \dfrac{1}{\pi^\pi}$,

$n > 72.941232$.

Thus, the least value of $n$ is $73$. A1

[2]

**2.** (a) $f'(x)$

$= (e^x)(1-x)^n + (e^x)(n)(1-x)^{n-1}(-1)$ A1

$= e^x(1-x)^{n-1}\left[(1-x)+n(-1)\right]$

$= e^x(1-x)^{n-1}(1-x-n)$ A1

$e^x > 0$, $(1-x)^{n-1} > 0$ and $1-x-n < 0$ for $n > 0$. R1

$\therefore f'(x) < 0$

Thus, $f(x)$ is decreasing in $0 < x < 1$ for $n > 0$. AG

[3]

(b) $f(0) = 1$ and $f(1) = 0$. R1

Also, $f(x)$ is decreasing in $0 < x < 1$.

Therefore, the area under the graph of $f(x)$ is positive, and is smaller than the area of the square of length $1$. R1

Thus, $0 < I(n) < 1$ for $n > 0$. AG

[2]

(c) (i) $I(0)$

$= \int_0^1 e^x(1-x)^0 \mathrm{d}x$ M1

$= \int_0^1 e^x \mathrm{d}x$

$= \left[e^x\right]_0^1$ A1

$= e^1 - e^0$

$= e - 1$ AG

(ii) $I(1)$

$= \int_0^1 e^x (1-x)^1 \mathrm{d}x$

Let $\theta = e^x$. (M1) for valid approach

$\frac{\mathrm{d}\theta}{\mathrm{d}x} = e^x$

$\therefore I(1)$

$= \int_0^1 (1-x) \cdot \frac{\mathrm{d}(e^x)}{\mathrm{d}x} \mathrm{d}x$

$= \left[ (1-x)e^x \right]_0^1 - \int_0^1 e^x \cdot \frac{\mathrm{d}(1-x)}{\mathrm{d}x} \mathrm{d}x$ A1

$= \left[ (1-x)e^x \right]_0^1 - \int_0^1 e^x (-1) \mathrm{d}x$ A1

$= \left[ (1-x)e^x \right]_0^1 + \int_0^1 e^x \mathrm{d}x$

$= \left[ (1-x)e^x \right]_0^1 + e - 1$ A1

$= ((1-1)e^1 - (1-0)e^0) + e - 1$

$= (0-1) + e - 1$

$= e - 2$ A1

(iii) $I(2)$

$= \int_0^1 e^x (1-x)^2 \mathrm{d}x$

Let $\theta = e^x$. (M1) for valid approach

$\frac{\mathrm{d}\theta}{\mathrm{d}x} = e^x$

$\therefore I(2)$

$= \int_0^1 (1-x)^2 \cdot \frac{\mathrm{d}(e^x)}{\mathrm{d}x} \mathrm{d}x$

$= \left[(1-x)^2 e^x\right]_0^1 - \int_0^1 e^x \cdot \frac{\mathrm{d}((1-x)^2)}{\mathrm{d}x} \mathrm{d}x$ A1

$= \left[(1-x)^2 e^x\right]_0^1 - \int_0^1 e^x \cdot 2(1-x)(-1) \mathrm{d}x$ A1

$= \left[(1-x)^2 e^x\right]_0^1 + 2\int_0^1 e^x (1-x) \mathrm{d}x$

$= \left[(1-x)^2 e^x\right]_0^1 + 2I(1)$ A1

$= \left[(1-x)^2 e^x\right]_0^1 + 2(e-2)$

$= ((1-1)^2 e^1 - (1-0)^2 e^0) + 2(e-2)$

$= (0-1) + 2e - 4$

$= 2e - 5$ A1

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(d) $I(n)$

$= \int_0^1 e^x (1-x)^n \mathrm{d}x$

Let $\theta = e^x$. M1

$\frac{\mathrm{d}\theta}{\mathrm{d}x} = e^x$

$\therefore I(n)$

$= \int_0^1 (1-x)^n \cdot \frac{\mathrm{d}(e^x)}{\mathrm{d}x} \mathrm{d}x$

$= \left[ (1-x)^n e^x \right]_0^1 - \int_0^1 e^x \cdot \frac{\mathrm{d}((1-x)^n)}{\mathrm{d}x} \mathrm{d}x$ A1

$= \left[ (1-x)^n e^x \right]_0^1 - \int_0^1 e^x \cdot n(1-x)^{n-1}(-1)\mathrm{d}x$ A1

$= \left[ (1-x)^n e^x \right]_0^1 + n\int_0^1 e^x (1-x)^{n-1} \mathrm{d}x$

$= \left[ (1-x)^n e^x \right]_0^1 + nI(n-1)$ A1

$= ((1-1)^n e^1 - (1-0)^n e^0) + nI(n-1)$

$= -1 + nI(n-1)$ A1

Thus, $I(n) = nI(n-1) - 1$ for $n > 0$. AG

[5]

(e) $I(n)$

$= nI(n-1) - 1$

$= n((n-1)I(n-2) - 1) - 1$ M1

$= n(n-1)I(n-2) - n - 1$

$= n(n-1)((n-2)I(n-3) - 1) - n - 1$ M1

$= n(n-1)(n-2)I(n-3) - n(n-1) - n - 1$

$= \cdots$

$= n(n-1)(n-2)\cdots(2)(1)I(0)$

$- n(n-1)(n-2)\cdots(2)$ A1

$- \cdots - n(n-1)(n-2) - n(n-1) - n - 1$

$$= n! \left[ \begin{array}{l} I(0) - \frac{1}{1!} - \cdots \\ - \frac{1}{(n-3)!} - \frac{1}{(n-2)!} - \frac{1}{(n-1)!} - \frac{1}{n!} \end{array} \right]$$ M1A1

$$= n! \left[ e - 1 - \left( \frac{1}{1!} + \cdots + \frac{1}{(n-2)!} + \frac{1}{(n-1)!} + \frac{1}{n!} \right) \right]$$

$\therefore I(n) = n! \left[ e - 1 - \sum_{r=1}^{n} \frac{1}{r!} \right]$ AG

[5]

(f) $e$ A1

[1]