AA HL Practice Set 1 Paper 3 Solution

1. (a) (i) $\frac{2\pi}{3}$ A1

(ii) A_1

$= \pi(1)^2 - 3\left(\frac{1}{2}(1)^2 \sin\frac{2\pi}{3}\right)$ M1A1

$= \pi - 3\left(\frac{1}{2}\sin\frac{2\pi}{3}\right)$

$= \pi - \frac{3}{2}\sin\frac{2\pi}{3}$ A1

$= \frac{3}{2}\left(\frac{2\pi}{3} - \sin\frac{2\pi}{3}\right)$

$= \left(\frac{1}{2}+1\right)\left(\frac{2\pi}{3} - \sin\frac{2\pi}{3}\right)$ AG

[4]

(b) (i) $\frac{\pi}{3}$ A1

(ii) $\frac{1}{2}\sin\frac{\pi}{3}$ A1

(iii) A_2

$= \pi(1)^2 - \frac{1}{2}\sin\frac{2\pi}{3} - 4\left(\frac{1}{2}\sin\frac{\pi}{3}\right)$ M1A1

$= \pi - \frac{1}{2}\sin\frac{2\pi}{3} - 2\sin\frac{\pi}{3}$

$= \frac{\pi}{3} - \frac{1}{2}\sin\frac{2\pi}{3} + \frac{2\pi}{3} - 2\sin\frac{\pi}{3}$ M1

$= \frac{1}{2}\left(\frac{2\pi}{3} - \sin\frac{2\pi}{3}\right) + 2\left(\frac{\pi}{3} - \sin\frac{\pi}{3}\right)$ AG

[5]

(c) (i) $Q_2\hat{O}Q$

$= \frac{2\pi}{3} \div 3$ (M1) for valid approach

$= \frac{2\pi}{9}$ A1

(ii) A_3

$= \pi(1)^2 - \frac{1}{2}\sin\frac{2\pi}{3} - 6\left(\frac{1}{2}\sin\frac{2\pi}{9}\right)$ M1A1

$= \pi - \frac{1}{2}\sin\frac{2\pi}{3} - 3\sin\frac{2\pi}{9}$

$= \frac{\pi}{3} - \frac{1}{2}\sin\frac{2\pi}{3} + \frac{2\pi}{3} - 3\sin\frac{2\pi}{9}$ M1

$= \frac{1}{2}\left(\frac{2\pi}{3} - \sin\frac{2\pi}{3}\right) + 3\left(\frac{2\pi}{9} - \sin\frac{2\pi}{9}\right)$ A1

[6]

(d) (i) A_n

$= \pi(1)^2 - \frac{1}{2}\sin\frac{2\pi}{3} - 2n\left(\frac{1}{2}\sin\left(\frac{2\pi}{3} \div n\right)\right)$ M1A1

$= \pi - \frac{1}{2}\sin\frac{2\pi}{3} - n\sin\frac{2\pi}{3n}$

$= \frac{\pi}{3} - \frac{1}{2}\sin\frac{2\pi}{3} + \frac{2\pi}{3} - n\sin\frac{2\pi}{3n}$ M1

$= \frac{1}{2}\left(\frac{2\pi}{3} - \sin\frac{2\pi}{3}\right) + n\left(\frac{2\pi}{3n} - \sin\frac{2\pi}{3n}\right)$ A1

$\therefore f(n) = \frac{2\pi}{3n} - \sin\frac{2\pi}{3n}$ A1

(ii) $f(n)$ represents the double of the area of the segment of the sector POQ_1. A1

[6]

(e) $\lim_{n\to\infty} f(n)$

$= \lim_{n\to\infty}\left(\frac{2\pi}{3n} - \sin\frac{2\pi}{3n}\right)$

$= \frac{2\pi}{3}\lim_{n\to\infty}\frac{1}{n} - \lim_{n\to\infty}\sin\frac{2\pi}{3n}$ M1

$= \frac{2\pi}{3}\lim_{n\to\infty}\frac{1}{n} - \sin\left(\frac{2\pi}{3}\lim_{n\to\infty}\frac{1}{n}\right)$

$= \frac{2\pi}{3}(0) - \sin\left(\frac{2\pi}{3}(0)\right)$

$= 0$ A1

[2]

(f) (i) $\frac{3}{2}\left(\frac{2\pi}{3} - \sin\frac{2\pi}{3}\right)$ A2

(ii) The maximum possible value of v

$= \lim_{n\to\infty} A_n$ M1

$= \lim_{n\to\infty}\left(\frac{1}{2}\left(\frac{2\pi}{3} - \sin\frac{2\pi}{3}\right) + n\cdot f(n)\right)$

$= \frac{1}{2}\left(\frac{2\pi}{3} - \sin\frac{2\pi}{3}\right)$ A1

[4]

2. (a) (i) $w^2 - w + 1 = 0$

$$w = \frac{-(-1) \pm \sqrt{(-1)^2 - 4(1)(1)}}{2(1)}$$ (A1) for substitution

$$w = \frac{1 \pm \sqrt{-3}}{2}$$

$$w = \frac{1 \pm \sqrt{3}\mathrm{i}}{2}$$

$$w = \frac{1}{2} \pm \frac{\sqrt{3}}{2}\mathrm{i}$$

$w = \cos\frac{\pi}{3} + \mathrm{i}\sin\frac{\pi}{3}$ or

$w = \cos\left(-\frac{\pi}{3}\right) + \mathrm{i}\sin\left(-\frac{\pi}{3}\right)$ A2

(ii) $u^4 - u^2 + 1 = 0$

$(u^2)^2 - u^2 + 1 = 0$ M1

$u^2 = \cos\frac{\pi}{3} + i\sin\frac{\pi}{3}$ or

$u^2 = \cos\left(-\frac{\pi}{3}\right) + i\sin\left(-\frac{\pi}{3}\right)$

$u = \cos\left(\frac{\frac{\pi}{3} + 2\pi k}{2}\right) + i\sin\left(\frac{\frac{\pi}{3} + 2\pi k}{2}\right)$ or

$u = \cos\left(\frac{-\frac{\pi}{3} + 2\pi k}{2}\right) + i\sin\left(\frac{-\frac{\pi}{3} + 2\pi k}{2}\right)$

$(k = 0, 1)$ A1

$u = \cos\frac{\pi}{6} + i\sin\frac{\pi}{6}$, $u = \cos\frac{7\pi}{6} + i\sin\frac{7\pi}{6}$,

$u = \cos\left(-\frac{\pi}{6}\right) + i\sin\left(-\frac{\pi}{6}\right)$ or

$u = \cos\frac{5\pi}{6} + i\sin\frac{5\pi}{6}$ A2

Thus, the required roots are

$\cos\left(-\frac{5\pi}{6}\right) + i\sin\left(-\frac{5\pi}{6}\right)$,

$\cos\left(-\frac{\pi}{6}\right) + i\sin\left(-\frac{\pi}{6}\right)$, $\cos\frac{\pi}{6} + i\sin\frac{\pi}{6}$ and

$\cos\frac{5\pi}{6} + i\sin\frac{5\pi}{6}$. AG

(iii) $z^{2n}-z^n+1=0$

$(z^n)^2-z^n+1=0$

$z^n=\cos\frac{\pi}{3}+i\sin\frac{\pi}{3}$ or

$z^n=\cos\left(-\frac{\pi}{3}\right)+i\sin\left(-\frac{\pi}{3}\right)$

$z=\cos\left(\frac{\frac{\pi}{3}+2\pi k}{n}\right)+i\sin\left(\frac{\frac{\pi}{3}+2\pi k}{n}\right)$ or

$z=\cos\left(\frac{-\frac{\pi}{3}+2\pi k}{n}\right)+i\sin\left(\frac{-\frac{\pi}{3}+2\pi k}{n}\right)$

$(k=0,1,2,\cdots,n-1)$) A1

$z=\cos\left(\frac{\pi}{3n}+\frac{2\pi}{n}k\right)+i\sin\left(\frac{\pi}{3n}+\frac{2\pi}{n}k\right)$ or

$z=\cos\left(-\frac{\pi}{3n}+\frac{2\pi}{n}k\right)+i\sin\left(-\frac{\pi}{3n}+\frac{2\pi}{n}k\right)$

$(k=0,1,2,\cdots,n-1)$)

$z=\cos\frac{\pi+6\pi k}{3n}+i\sin\frac{\pi+6\pi k}{3n}$ or

$z=\cos\frac{-\pi+6\pi k}{3n}+i\sin\frac{-\pi+6\pi k}{3n}$

$(k=0,1,2,\cdots,n-1)$) A1

Thus, the required roots are

$\cos\left(-\frac{\pi}{3n}\right)+i\sin\left(-\frac{\pi}{3n}\right)$, $\cos\frac{\pi}{3n}+i\sin\frac{\pi}{3n}$,

$\cos\frac{5\pi}{3n}+i\sin\frac{5\pi}{3n}$, $\cos\frac{7\pi}{3n}+i\sin\frac{7\pi}{3n}$, $\cdots$,

$\cos\frac{(6n-7)\pi}{3n}+i\sin\frac{(6n-7)\pi}{3n}$ and

$\cos\frac{(6n-5)\pi}{3n}+i\sin\frac{(6n-5)\pi}{3n}$. A3

[12]

(b) (i) $(z-(\cos\theta+\mathrm{i}\sin\theta))(z-(\cos(-\theta)+\mathrm{i}\sin(-\theta)))$

$=(z-\cos\theta-\mathrm{i}\sin\theta)(z-\cos\theta+\mathrm{i}\sin\theta)$

$=z^2-z\cos\theta+\mathrm{i}z\sin\theta-z\cos\theta+\cos^2\theta$

$-\mathrm{i}\sin\theta\cos\theta-\mathrm{i}z\sin\theta+\mathrm{i}\sin\theta\cos\theta+\sin^2\theta$ M1

$=z^2-2z\cos\theta+1$ A1

(ii) u^4-u^2+1

$=\left(u-\left(\cos\frac{\pi}{6}+\mathrm{i}\sin\frac{\pi}{6}\right)\right)$

$\left(u-\left(\cos\left(-\frac{\pi}{6}\right)+\mathrm{i}\sin\left(-\frac{\pi}{6}\right)\right)\right)$

$\left(u-\left(\cos\frac{5\pi}{6}+\mathrm{i}\sin\frac{5\pi}{6}\right)\right)$ M1A1

$\left(u-\left(\cos\left(-\frac{5\pi}{6}\right)+\mathrm{i}\sin\left(-\frac{5\pi}{6}\right)\right)\right)$

$=\left(u^2-2u\cos\frac{\pi}{6}+1\right)\left(u^2-2u\cos\frac{5\pi}{6}+1\right)$ AG

(iii) The roots of the equation $z^6-z^3+1=0$

are $\mathrm{cis}\frac{\pi}{9}$, $\mathrm{cis}\left(-\frac{\pi}{9}\right)$, $\mathrm{cis}\frac{5\pi}{9}$, $\mathrm{cis}\left(-\frac{5\pi}{9}\right)$,

$\mathrm{cis}\frac{7\pi}{9}$ and $\mathrm{cis}\left(-\frac{7\pi}{9}\right)$. (A1) for correct values

z^6-z^3+1

$=\left(z-\mathrm{cis}\frac{\pi}{9}\right)\left(z-\mathrm{cis}\left(-\frac{\pi}{9}\right)\right)\left(z-\mathrm{cis}\frac{5\pi}{9}\right)$

$\left(z-\mathrm{cis}\left(-\frac{5\pi}{9}\right)\right)\left(z-\mathrm{cis}\frac{7\pi}{9}\right)$ A1

$\left(z-\mathrm{cis}\left(-\frac{7\pi}{9}\right)\right)$

$=\left(z^2-2z\cos\frac{\pi}{9}+1\right)\left(z^2-2z\cos\frac{5\pi}{9}+1\right)$

$\left(z^2-2z\cos\frac{7\pi}{9}+1\right)$ A1

(iv) $z^{2n} - z^{n} + 1 = 0$

$$= \left(z^2 - 2z\cos\frac{\pi}{3n} + 1\right)\left(z^2 - 2z\cos\frac{5\pi}{3n} + 1\right)$$

$$\left(z^2 - 2z\cos\frac{7\pi}{3n} + 1\right)\cdots$$

A2

$$\left(z^2 - 2z\cos\left(\pi - \frac{5\pi}{3n}\right) + 1\right)$$

$$\left(z^2 - 2z\cos\left(\pi - \frac{\pi}{3n}\right) + 1\right)$$

[9]

(c) $u^4 - u^2 + 1 = \left(u^2 - 2u\cos\frac{\pi}{6} + 1\right)\left(u^2 - 2u\cos\frac{5\pi}{6} + 1\right)$

When $u = \mathrm{i}$,

$\mathrm{i}^4 - \mathrm{i}^2 + 1 = \left(\mathrm{i}^2 - 2\mathrm{i}\cos\frac{\pi}{6} + 1\right)\left(\mathrm{i}^2 - 2\mathrm{i}\cos\frac{5\pi}{6} + 1\right)$ M1

$1 - (-1) + 1 = \left(-1 - 2\mathrm{i}\cos\frac{\pi}{6} + 1\right)\left(-1 - 2\mathrm{i}\cos\frac{5\pi}{6} + 1\right)$ A1

$3 = \left(-2\mathrm{i}\cos\frac{\pi}{6}\right)\left(-2\mathrm{i}\cos\frac{5\pi}{6}\right)$

$3 = 4\mathrm{i}^2\cos\frac{\pi}{6}\cos\frac{5\pi}{6}$ A1

$3 = -4\cos\frac{\pi}{6}\cos\frac{5\pi}{6}$

$\cos\frac{\pi}{6}\cos\frac{5\pi}{6} = -\frac{3}{4}$ AG

[3]

(d) $z^6 - z^3 + 1 = \left(z^2 - 2z\cos\frac{\pi}{9} + 1\right)\left(z^2 - 2z\cos\frac{5\pi}{9} + 1\right)$

$\left(z^2 - 2z\cos\frac{7\pi}{9} + 1\right)$

When $z = \mathrm{i}$,

$\mathrm{i}^6 - \mathrm{i}^3 + 1 = \left(\mathrm{i}^2 - 2\mathrm{i}\cos\frac{\pi}{9} + 1\right)\left(\mathrm{i}^2 - 2\mathrm{i}\cos\frac{5\pi}{9} + 1\right)$

$\left(\mathrm{i}^2 - 2\mathrm{i}\cos\frac{7\pi}{9} + 1\right)$ (M1) for valid approach

$-1 - (-\mathrm{i}) + 1 = \left(-1 - 2\mathrm{i}\cos\frac{\pi}{9} + 1\right)$

$\left(-1 - 2\mathrm{i}\cos\frac{5\pi}{9} + 1\right)\left(-1 - 2\mathrm{i}\cos\frac{7\pi}{9} + 1\right)$ A1

$\mathrm{i} = \left(-2\mathrm{i}\cos\frac{\pi}{9}\right)\left(-2\mathrm{i}\cos\frac{5\pi}{9}\right)\left(-2\mathrm{i}\cos\frac{7\pi}{9}\right)$

$\mathrm{i} = -8\mathrm{i}^3\cos\frac{\pi}{9}\cos\frac{5\pi}{9}\cos\frac{7\pi}{9}$ A1

$\mathrm{i} = 8\mathrm{i}\cos\frac{\pi}{9}\cos\frac{5\pi}{9}\cos\frac{7\pi}{9}$

$\cos\frac{\pi}{9}\cos\frac{5\pi}{9}\cos\frac{7\pi}{9} = \frac{1}{8}$ A1

[4]