AA HL Practice Set 2 Paper 1 Solution

Section A

1. (a) (i) 7 A1

(ii) 1 A1

[2]

(b) $(f \circ g)(x)$

$= (g(x))^2$ (A1) for substitution

$= (3-4x)^2$

$= 9-24x+16x^2$ A1

[2]

(c) $y = 3-4x$

$\Rightarrow x = 3-4y$ (A1) for correct approach

$4y = 3-x$

$y = \dfrac{3-x}{4}$

$\therefore g^{-1}(x) = \dfrac{3-x}{4}$ A1

[2]

2. (a) R.H.S.

$= \frac{1\times 49}{1\times 49}+\frac{2\times 7}{7\times 7}+\frac{5}{49}$ M1

$= \frac{49+14+5}{49}$ A1

$= \frac{68}{49} = \text{L.H.S.}$

$\therefore \frac{68}{49} = 1+\frac{2}{7}+\frac{5}{49}$ AG

[2]

(b) R.H.S.

$= \frac{1\times (m+2)^2}{1\times (m+2)^2}+\frac{2\times (m+2)}{(m+2)\times (m+2)}+\frac{5}{(m+2)^2}$ M1

$= \frac{(m^2+4m+4)+(2m+4)+5}{(m+2)^2}$ M1A1

$= \frac{m^2+6m+9+4}{(m+2)^2}$

$= \frac{(m+3)^2+4}{(m+2)^2} = \text{L.H.S.}$

$\therefore \frac{(m+3)^2+4}{(m+2)^2} \equiv 1+\frac{2}{m+2}+\frac{5}{(m+2)^2}$ for $m \neq -2$ AG

[3]

3. $P(2)=0$

$a(2)^3+b(2)^2-10(2)+24=0$ (M1) for factor theorem

$4b=-4-8a$

$b=-1-2a$ A1

$P(-3)=0$

$a(-3)^3+b(-3)^2-10(-3)+24=0$

$-27a+9b+30+24=0$

$\therefore -27a+9(-1-2a)+30+24=0$ (M1) for substitution

$-27a-9-18a+30+24=0$

$-45a=-45$

$a=1$ A1

$b=-1-2(1)$

$b=-3$ A1

[5]

4. (a) The discriminant of $f(x)$

$=b^2-4ac$

$=(8-p)^2-4\left(1+2p-\frac{3}{8}p^2\right)(-2)$ M1A1

$=64-16p+p^2+8+16p-3p^2$ A1

$=72-2p^2$ AG

[3]

(b) $f(x)=0$ has two equal roots

$\therefore 72-2p^2=0$ (M1) for setting equation

$2p^2=72$

$p^2=36$

$p=-6$ or $p=6$ A2

[3]

(c) $p=6$

$\therefore\left(1+2(6)-\frac{3}{8}(6)^2\right)x^2+(8-6)x-2=0$ (M1) for setting equation

$-\frac{1}{2}x^2+2x-2=0$

$x^2-4x+4=0$

$(x-2)^2=0$

$x=2$ A1

[2]

5. $9\log_{27}(x+1)=1+\log_3(3+x+x^2)$

$\frac{9\log_3(x+1)}{\log_3 27}=\log_3 3+\log_3(3+x+x^2)$ (M1)(A1) for change of base

$\frac{9\log_3(x+1)}{3}=\log_3 3(3+x+x^2)$ (A1) for correct approach

$3\log_3(x+1)=\log_3 3(3+x+x^2)$

$\log_3(x+1)^3=\log_3 3(3+x+x^2)$ A1

$\therefore (x+1)^3=3(3+x+x^2)$ M1

$x^3+3x^2+3x+1=9+3x+3x^2$

$x^3=8$ A1

$x=\sqrt[3]{8}$

$x=2$ A1

[7]

6. (a) $r = \dfrac{20\cos^4\alpha}{30\cos^2\alpha}$ (M1) for valid approach

$r = \dfrac{2}{3}\cos^2\alpha$ A1

[2]

(b) $\pi \le \alpha \le \dfrac{4}{3}\pi$

$\therefore \cos\pi \le \cos\alpha \le \cos\dfrac{4}{3}\pi$ (M1) for valid approach

$-1 \le \cos\alpha \le -\dfrac{1}{2}$

$\dfrac{1}{4} \le \cos^2\alpha \le 1$

$\dfrac{1}{6} \le \dfrac{2}{3}\cos^2\alpha \le \dfrac{2}{3}$

$\therefore \dfrac{1}{6} \le r \le \dfrac{2}{3}$ A1

[2]

(c) $S_\infty = \dfrac{30\cos^2\alpha}{1-\dfrac{2}{3}\cos^2\alpha}$ A1

$S_\infty = \dfrac{30\cos^2\alpha}{\sin^2\alpha + \cos^2\alpha - \dfrac{2}{3}\cos^2\alpha}$ M1

$S_\infty = \dfrac{30\cos^2\alpha}{\sin^2\alpha + \dfrac{1}{3}\cos^2\alpha}$ A1

$S_\infty = \dfrac{30}{\tan^2\alpha + \dfrac{1}{3}}$ A1

$S_\infty = \dfrac{90}{3\tan^2\alpha + 1}$ AG

[4]

7. When $n=1$,

$\text{L.H.S.}=1^2$

$\text{L.H.S.}=1$

$\text{R.H.S.}=\frac{4}{3}(1)^3$

$\text{R.H.S.}=\frac{4}{3}$

Thus, the statement is true when $n=1$. R1

Assume that the statement is true when $n=k$. M1

$1^2+2^2+\ldots+k^2\le\frac{4}{3}k^3$

When $n=k+1$,

$1^2+2^2+\ldots+k^2+(k+1)^2$

$\le\frac{4}{3}k^3+(k+1)^2$ M1A1

$=\frac{4k^3+3(k^2+2k+1)}{3}$ A1

$=\frac{4k^3+3k^2+6k+3}{3}$

$\le\frac{4k^3+12k^2+12k+4}{3}$ A1

$=\frac{4(k^3+3k^2+3k+1)}{3}$

$=\frac{4}{3}(k+1)^3$

Thus, the statement is true when $n=k+1$.

Therefore, the statement is true for all $n\in\mathbb{Z}^+$. R1

[7]

8. $\lim_{x \to 0} \frac{1-e^{x^2}}{1-\sec x}$

$= \lim_{x \to 0} \frac{0-(e^{x^2})(2x)}{-\sec x \tan x} \left(\because \frac{0}{0}\right)$ M1A2

$= \lim_{x \to 0} \frac{2xe^{x^2}}{\sec x \tan x}$

$= \lim_{x \to 0} \frac{(2)(e^{x^2})+(2x)(e^{x^2})(2x)}{(\sec x \tan x)(\tan x)+(\sec x)(\sec^2 x)} \left(\because \frac{0}{0}\right)$ A2

$= \lim_{x \to 0} \frac{2e^{x^2}+4x^2e^{x^2}}{\sec x \tan^2 x + \sec^3 x}$

$= \frac{2e^0+4(0)^2e^0}{\sec 0 \tan^2 0 + \sec^3 0}$ M1

$= \frac{2+0}{0+1}$

$= 2$ A1

[7]

9. (a) $-\pi a$ A1

[1]

(b) $\int_{-\pi a}^{a} |x| \, dx = 1$ A1

$\int_{-\pi a}^{0} -x dx + \int_{0}^{a} x dx = 1$

$\left[-\frac{1}{2}x^2\right]_{-\pi a}^{0} + \left[\frac{1}{2}x^2\right]_{0}^{a} = 1$ A1

$\left(0-\left(-\frac{1}{2}\pi^2a^2\right)\right)+\left(\frac{1}{2}a^2-0\right)=1$

$\frac{1}{2}\pi^2a^2+\frac{1}{2}a^2=1$ M1

$a^2(\pi^2+1)=2$

$a^2=\frac{2}{\pi^2+1}$ A1

$a=-\sqrt{\frac{2}{\pi^2+1}}$ (*Rejected*) or $a=\sqrt{\frac{2}{\pi^2+1}}$

Thus, $a=\sqrt{\frac{2}{\pi^2+1}}$. AG

[4]

Section B

10. (a)

$2r+h=20$	(A1) for correct approach
$2r=20-h$	
$r=10-\frac{1}{2}h$	A1
	[2]

(b)

$V=\pi r^2 h$	
$V=\pi\left(10-\frac{1}{2}h\right)^2 h$	(A1) for substitution
$V=100\pi h-10\pi h^2+\frac{1}{4}\pi h^3$	A1
	[2]

(c)

$Q=(3)(2\pi rh)+(4)(\pi r^2)$	M1A1
$Q=6\pi\left(10-\frac{1}{2}h\right)h+4\pi\left(10-\frac{1}{2}h\right)^2$	M1
$Q=60\pi h-3\pi h^2+400\pi-40\pi h+\pi h^2$	A1
$Q=400\pi+20\pi h-2\pi h^2$	
$Q=2\pi(200+10h-h^2)$	AG
	[4]

(d)

$\frac{\mathrm{d}Q}{\mathrm{d}h}=2\pi(0+10(1)-2h)$	(A1) for correct derivatives
$\frac{\mathrm{d}Q}{\mathrm{d}h}=4\pi(5-h)$	A1
$\frac{\mathrm{d}Q}{\mathrm{d}h}=0$	(M1) for setting equation
$\therefore 4\pi(5-h)=0$	A1
$h=5$	A1
The maximum value of Q	
$=2\pi(200+10(5)-(5)^2)$	(M1) for substitution
$=450\pi$	A1
	[7]

11. (a) $\overrightarrow{BD} = \begin{pmatrix} -9 \\ 0 \\ 0 \end{pmatrix} - \begin{pmatrix} 0 \\ 9 \\ -9 \end{pmatrix}$

$\overrightarrow{BD} = \begin{pmatrix} -9 \\ -9 \\ 9 \end{pmatrix}$ A1

The vector equation of BD:

$\mathbf{r} = \begin{pmatrix} 0 \\ 9 \\ -9 \end{pmatrix} + t\begin{pmatrix} -9 \\ -9 \\ 9 \end{pmatrix}$

$\begin{cases} x = -9t \\ y = 9 - 9t \\ z = -9 + 9t \end{cases}$ A1

$\overrightarrow{CE} = \begin{pmatrix} -9t \\ 9-9t \\ -9+9t \end{pmatrix} - \begin{pmatrix} 0 \\ 0 \\ -9 \end{pmatrix}$

$\overrightarrow{CE} = \begin{pmatrix} -9t \\ 9-9t \\ 9t \end{pmatrix}$ A1

$\overrightarrow{CE} \cdot \overrightarrow{BD} = 0$

$\therefore (-9t)(-9) + (9-9t)(-9) + (9t)(9) = 0$ M1

$81t - 81 + 81t + 81t = 0$

$243t = 81$

$t = \frac{1}{3}$ A1

$\therefore \begin{cases} x = -9\left(\frac{1}{3}\right) = -3 \\ y = 9 - 9\left(\frac{1}{3}\right) = 6 \\ z = -9 + 9\left(\frac{1}{3}\right) = -6 \end{cases}$ M1

Therefore, the coordinates of E are $(-3, 6, -6)$.AG

[6]

(b) $\overrightarrow{BA} = \begin{pmatrix} 0 \\ 9 \\ 0 \end{pmatrix} - \begin{pmatrix} 0 \\ 9 \\ -9 \end{pmatrix}$

$\overrightarrow{BA} = \begin{pmatrix} 0 \\ 0 \\ 9 \end{pmatrix}$ (A1) for correct values

$\overrightarrow{BC} = \begin{pmatrix} 0 \\ 0 \\ -9 \end{pmatrix} - \begin{pmatrix} 0 \\ 9 \\ -9 \end{pmatrix}$

$\overrightarrow{BC} = \begin{pmatrix} 0 \\ -9 \\ 0 \end{pmatrix}$ (A1) for correct values

$\mathbf{n}_1 = \overrightarrow{BA} \times \overrightarrow{BD}$ (M1) for valid approach

$\mathbf{n}_1 = \begin{pmatrix} 0 \\ 0 \\ 9 \end{pmatrix} \times \begin{pmatrix} -9 \\ -9 \\ 9 \end{pmatrix}$

$\mathbf{n}_1 = \begin{pmatrix} (0)(9) - (9)(-9) \\ (9)(-9) - (0)(9) \\ (0)(-9) - (0)(-9) \end{pmatrix}$

$\mathbf{n}_1 = \begin{pmatrix} 81 \\ -81 \\ 0 \end{pmatrix}$ A1

$\mathbf{n}_2 = \overrightarrow{BC} \times \overrightarrow{BD}$ (M1) for valid approach

$\mathbf{n}_2 = \begin{pmatrix} 0 \\ -9 \\ 0 \end{pmatrix} \times \begin{pmatrix} -9 \\ -9 \\ 9 \end{pmatrix}$

$\mathbf{n}_2 = \begin{pmatrix} (-9)(9) - (0)(-9) \\ (0)(-9) - (0)(9) \\ (0)(-9) - (-9)(-9) \end{pmatrix}$

$\mathbf{n}_2 = \begin{pmatrix} -81 \\ 0 \\ -81 \end{pmatrix}$ A1

$\mathbf{n}_1 \cdot \mathbf{n}_2 = |\mathbf{n}_1||\mathbf{n}_2|\cos\theta$ (M1) for valid approach

$(81)(-81)+(-81)(0)+(0)(-81)$

$=(\sqrt{81^2+(-81)^2})(\sqrt{(-81)^2+(-81)^2})\cos\theta$ A1

$-81^2=2(81)^2\cos\theta$

$\cos\theta=-\frac{1}{2}$

$\theta=120^\circ$ A1

[9]

(c) The area of OABC

$=(\text{OA})(\text{OC})$

$=(9)(9)$

$=81$ (A1) for correct value

$\therefore\frac{1}{3}(81)(\text{OD})+\frac{1}{3}(81)(\text{OF})=783$ (M1) for setting equation

$\frac{1}{3}(81)(9)+\frac{1}{3}(81)(\text{OF})=783$

$27\text{OF}=540$

$\text{OF}=20$ A1

$\therefore\text{DF}=9+20$

$\text{DF}=29$ A1

[4]

12. (a)

$$\frac{da}{dt} - 2a^2 = 50$$

$$\frac{da}{dt} = 2a^2 + 50$$

$$\frac{da}{dt} = 2(a^2 + 25)$$

$$\frac{1}{a^2+25}\,da = 2dt$$ (M1) for valid approach

$$\int \frac{1}{a^2+25}\,da = \int 2dt$$ (A1) for correct approach

$$\int \frac{1}{a^2+5^2}\,da = \int 2dt$$

$$\frac{1}{5}\arctan\frac{a}{5} = 2t + C$$ A1

$$\arctan\frac{a}{5} = 10t + C$$

$$\frac{a}{5} = \tan(10t + C)$$

$$a = 5\tan(10t + C)$$ A1

$$5 = 5\tan(10(0) + C)$$ (M1) for substitution

$$1 = \tan C$$

$$\tan C = \tan\frac{\pi}{4}$$

$$C = \frac{\pi}{4}$$ (A1) for correct value

$$\therefore a = 5\tan\left(10t + \frac{\pi}{4}\right)$$ A1

[7]

(b)

$$\frac{dv}{dt} = 5\tan\left(10t + \frac{\pi}{4}\right)$$

$$dv = \frac{5\sin\left(10t + \frac{\pi}{4}\right)}{\cos\left(10t + \frac{\pi}{4}\right)}dt$$

$$\int dv = \int \frac{5\sin\left(10t + \frac{\pi}{4}\right)}{\cos\left(10t + \frac{\pi}{4}\right)}dt$$ (A1) for correct approach

Let $u = \cos\left(10t + \frac{\pi}{4}\right)$. (M1) for substitution

$$\frac{du}{dt} = -10\sin\left(10t + \frac{\pi}{4}\right) \Rightarrow 5\sin\left(10t + \frac{\pi}{4}\right)dt = -\frac{1}{2}du$$

$\therefore \int dv = -\frac{1}{2}\int \frac{1}{u}du$ (A1) for correct working

$v = -\frac{1}{2}\ln u + D$

$v = -\frac{1}{2}\ln\left|\cos\left(10t + \frac{\pi}{4}\right)\right| + D$ A1

$\ln 2^{\frac{1}{4}} = -\frac{1}{2}\ln\left|\cos\left(10(0) + \frac{\pi}{4}\right)\right| + D$ (M1) for substitution

$\frac{1}{4}\ln 2 = -\frac{1}{2}\ln\frac{\sqrt{2}}{2} + D$

$\frac{1}{4}\ln 2 = \frac{1}{4}\ln 2 + D$

$D = 0$ (A1) for correct value

$\therefore v = -\frac{1}{2}\ln\left|\cos\left(10t + \frac{\pi}{4}\right)\right|$ A1

[7]

(c) $v = -\frac{1}{2}\ln\left|\cos\left(10t + \frac{\pi}{4}\right)\right|$

$v = \frac{1}{2}\ln\left|\frac{1}{\cos\left(10t + \frac{\pi}{4}\right)}\right|$ M1

$v = \frac{1}{2}\ln\left|\sec\left(10t + \frac{\pi}{4}\right)\right|$ A1

$v = \frac{1}{4}\ln\left(\sec^2\left(10t + \frac{\pi}{4}\right)\right)$

$v = \frac{1}{4}\ln\left(\tan^2\left(10t + \frac{\pi}{4}\right) + 1\right)$ A1

$\therefore v = \frac{1}{4}\ln\left(\left(\frac{a}{5}\right)^2 + 1\right)$ A1

$v = \frac{1}{4}\ln\left(\frac{a^2}{25} + 1\right)$

$v = \frac{1}{4}\ln\left(\frac{a^2 + 25}{25}\right)$ AG

[4]